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On the Complexity of Orthogonal Compaction

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ABSTRACT

We consider three closely related optimization problems, arising from the graph drawing and the VLSI research areas, and conjectured to be NP-hard, and we prove that, in fact, they are NP-complete.

Starting from an orthogonal representation of a graph, i.e., a description of the shape of the edges that does not specify segment lengths or vertex positions, the three problems consist of providing an orthogonal grid drawing of it, while minimizing the area, the total edge length, or the maximum edge length, respectively.

This result confirms a long surviving conjecture of NP-hardness, justifies the research about applying sophisticated, yet possibly time consuming, techniques to obtain optimally compacted orthogonal grid drawings, discourages the quest for an optimally compacting polynomial-time algorithm, and opens the research about the approximability of the three problems by showing that they don't allow a polynomial-time approximation scheme.

1 Introduction

The orthogonal drawing standard is recognized to be suitable for several types of diagrams, including data flow diagrams, entity-relationship diagrams, state-transition charts, circuit schematics, and many others. Such diagrams are extensively used in real-life applications spanning from software engineering, to databases, real-time systems, and VLSI.

A well known approach to produce orthogonal drawings is the topology-shape-metric approach (see, for example, [12, 5, 1, 9, 3]), in which the graph drawing process is organized in three steps (see Fig. 1):

planarization step: determines the topology of the drawing, which is described by a planar embedding, i.e., the order of the edges around each vertex. The purpose of this step is to minimize edge crossings. Also, each crossing is replaced by a dummy vertex, so that the final topology is planar.

orthogonalization step: determines the shape of the drawing, in which vertices do not have coordinates and each edge is equipped with a list of angles, describing the bends featured by the orthogonal line representing the edge in the final drawing. The purpose of this step is the reduction of the total number of bends.

compaction step: determines the final coordinates of the vertices and bends. Also, at the end of this step the dummy vertices introduced in the planarization step are removed.

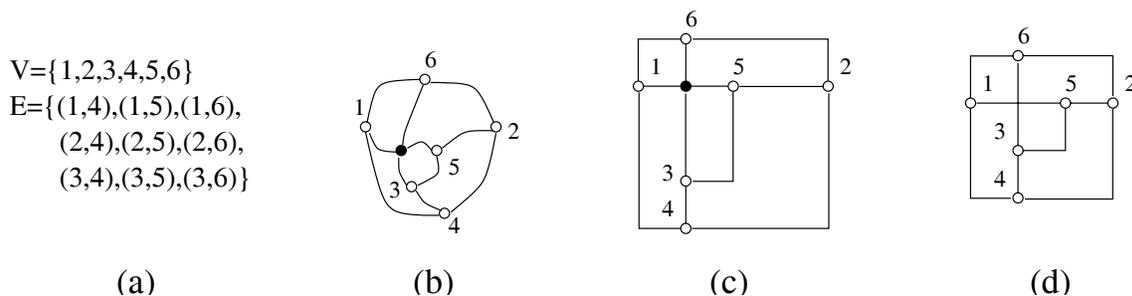


Figure 1: the three steps of the topology-shape-metric approach: (a) abstract description of a graph; (b) planar embedding produced by the planarization step; (c) orthogonal representation produced by the orthogonalization step; and (d) final drawing produced by the compaction step. The dummy vertex (black) is introduced by the planarization step and removed at the end of the compaction step.

The name of the last step originates from the fact that during this step an aesthetic measure between area, total edge length, or maximum edge length is hopefully minimized. The compaction problem is precisely the optimization problem consisting of minimizing one of the three measures just mentioned, while performing the compaction step: in particular we call Orthogonal Area Compaction (OAC), Orthogonal Total Edge Length Compaction (OTELC), and Orthogonal Maximum Edge Length Compaction (OMELC) the three problems, respectively.

Finding the intrinsic computational complexity of the compaction problem has been for a long time an elusive goal. Decades of intense research in the field of orthogonal

graph drawing have not affected our knowledge in this respect: the problem is mentioned as open in recent papers ([8]) as in founding ones ([13]). As far as we know, the only contribution to this subject is the early result contained in [4], where the trivial case of not connected graphs is demonstrated to be NP-complete.

The compaction problem has been one of the challenging tasks in the VLSI research field too, where the requirement of minimizing the size of a circuit layout while preserving its shape, led to formulations similar to those arising in the graph drawing area, although, for VLSI purposes, vertices are possibly replaced by squares and additional constraints (e.g. on the length of specific edges) are generally managed. Since several VLSI formulations, related with the compaction problem, are proved to be NP-hard [10], compacting orthogonal representations is widely believed to be an NP-hard problem too, and heuristics producing suboptimal solutions are applied in all practical cases.

A first strain of heuristics descend from the “rectangular refinement” approach proposed in [12], based on the fact that the compaction problem is tractable when all faces of the orthogonal representation are rectangular, and consisting of splitting the non rectangular faces into rectangles and removing the introduced edges after compaction. This approach may yield a linear time compaction step that minimizes the area, or an $O(n^{7/4} \log n)$ compaction step that minimizes the area and (secondarily) the total edge length [3].

Recently, the compaction step has been the subject of a renewed research interest. The problem of optimal compaction with respect to total edge length was approached with an ILP formulation in [8], relying on branch-and-cut or branch-and-bound techniques to find an optimal solution. Lately, a novel compaction method has been devised that optimizes with respect to the area (and, secondarily, total edge length) in polynomial time in the particular, though relatively frequent, case of turn-regular orthogonal representations ([2]). The latter approach gives raise to new heuristics based on a “turn regularization” rather than a “rectangular refinement” preprocessing step.

In this paper, by means of a reduction from the SAT problem, we prove that compacting an orthogonal representation of a connected graph, while minimizing an aesthetic measure between area, total edge length, and maximum edge length is an NP-complete problem. To accomplish this, after formally defining the three problems in Section 2, we introduce in Section 3 a class of orthogonal representations, that we call *sliding rectangles press gadgets*, admitting an exponential number of orthogonal grid drawings with minimum area, in all of which the basic blocks composing the gadget necessarily inherit the property of being themselves drawn with the minimum area. This property will be exploited in Section 4 to build a sliding rectangles press gadget corresponding to a formula ϕ of the SAT problem. We will prove the NP-hardness of the OAC problem by showing that such orthogonal representation admits exclusively the subset the orthogonal grid drawings, with a given goal area, corresponding to the truth assignments satisfying the formula ϕ .

In Section 5 the result of Section 4 is extended to the problems of compacting an orthogonal representation while minimizing the total edge length and the maximum edge length. In Section 6 and 7 the three problems are proved to be in NP, and in PTAS, respectively. Finally, Section 8 contains our conclusions.

2 Preliminaries

We assume familiarity with basic graph theoretic concepts and graph drawing terminology (see, e.g., [7] and [3], respectively) concerning planarity, planar graphs, and planar representations.

In a *planar orthogonal drawing* Γ of a graph G , vertices are placed on distinct points of the plane and edges are alternating sequences of horizontal and vertical segments, non intersecting except at edges common endpoints.

In what follows we will consider, without loss of generality, only orthogonal drawings with no bends, since each bend can be replaced by a dummy vertex of degree two. According to this assumption, our definition of orthogonal representation, although similar to the usual one (see, e.g., [12, 2, 3]), will be specialized for capturing the notion of “orthogonal shape” of a planar orthogonal graph without bends.

Let f be a face of an embedded 4-planar graph G . We always traverse the boundary of f in such a way that f is on the left, i.e., counterclockwise or clockwise, depending on whether f is an internal or external face, respectively.

Lets call Σ the set of labels $\{l, s, r, b\}$ (left, straight, right, and back). Given a planar orthogonal drawing Γ of G , we associate with a pair of consecutive edges (possibly coinciding) of a face f , a label λ with value l (s , r , b , respectively) if the angle formed by the two consecutive edges in the face f is 90° (180° , 270° , 360° , respectively).

An orthogonal representation of G describes an equivalence class of planar orthogonal drawings of G with “similar shape”, that is, with the same labels associated with the angles of G . More formally, an *orthogonal representation* of a graph G is an assignment of a label $\lambda \in \Sigma$ to each angle formed by a pair of consecutive edges around each vertex of G satisfying the following properties:

1. given a vertex v , the angles around v may be labeled (l, l, l, l) , (l, l, s) , (l, r) , (s, s) , or (b) , depending on their number.
2. denoted by l_f , r_f , and b_f the number of l , r , and b labeled angles around the boundary of face f , then $l_f - r_f - 2b_f = 4$, if f is an internal face, or -4 , if f is the external face.

An *orthogonal grid drawing* Γ of a graph G (without bends) is an orthogonal drawing such that vertex coordinates have integer values.

The *length* of an orthogonal grid drawing Γ with respect to the x axis is defined as the maximum difference between the x coordinates of its vertices, and is denoted by ℓ_x^Γ . The minimum ℓ_x^Γ for all orthogonal grid drawings of a given orthogonal representation is denoted ℓ_x . In the same way ℓ_y^Γ and ℓ_y are defined with respect to the y axis. A (w, h) -*compactable orthogonal representation* is an orthogonal representation that admits an orthogonal grid drawing Γ for which is both $\ell_x^\Gamma = \ell_x = w$ and $\ell_y^\Gamma = \ell_y = h$.

The *area* of an orthogonal grid drawing Γ is the product $\ell_x^\Gamma \cdot \ell_y^\Gamma$. The *total edge length* of an orthogonal grid drawing is the sum of the lengths of its edges. The *maximum edge length* of an orthogonal grid drawing Γ is the maximum value of all its edge lengths.

This paper is concerned with the complexity of producing an orthogonal grid drawing Γ starting from its orthogonal representation H while minimizing the area of the drawing, the total edge length, or the maximum edge length. The three minimization criteria are considered to have roughly the same aesthetic effect: that of reducing the size of

the drawing (or of part of it) and so improve its readability. However, conflicts between the three requirements (see Fig. 2) imply that they constitute different, although closely related, optimization problems.

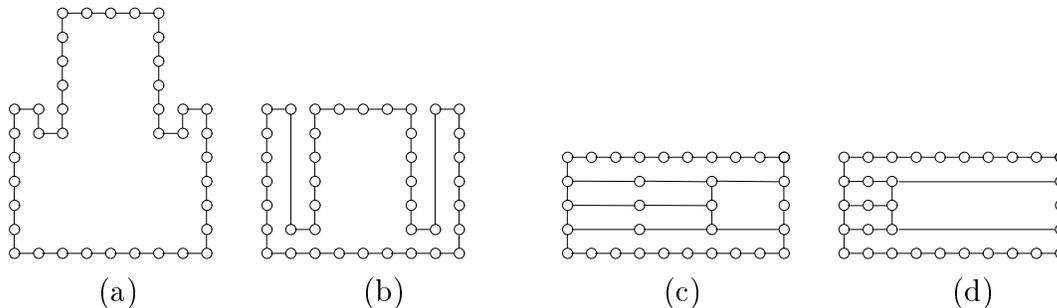


Figure 2: The orthogonal drawings (a) and (b) correspond to the same orthogonal representation, and show how the two requirements of minimum area and minimum total (or minimum maximum) edge length may not be satisfiable by a single drawing (observe that the graph is biconnected and its orthogonal representation is “turn-regular” as defined in [2]). The drawings (c) and (d) too correspond to a single orthogonal representation: (c) minimizes the maximum edge length and (d) the total edge length (observe that the graph is biconnected, its orthogonal representation is “turn-regular” and rectangular).

Following an usual technique (see, e.g., [6, 11]), rather than address directly the three optimization problems we will consider their corresponding decision versions that can be formally defined as follows:

Problem: **Orthogonal Area Compaction (OAC)**

Instance: An orthogonal representation H of a graph G and a constant K .

Question: Can integer coordinates be assigned to the vertices of G so that the **area** of the drawing is less or equal than the value of the constant K ?

Problem: **Orthogonal Total Edge Length Compaction (OTELC)**

Instance: An orthogonal representation H of a graph G and a constant K .

Question: Can integer coordinates be assigned to the vertices of G so that the **total edge length** of the drawing is less or equal than the value of the constant K ?

Problem: **Orthogonal Maximum Edge Length Compaction (OMELC)**

Instance: An orthogonal representation H of a graph G and a constant K .

Question: Can integer coordinates be assigned to the vertices of G so that the **maximum edge length** of the drawing is less or equal than the value of the constant K ?

We will show in the following sections that the three problems above are NP-hard and are in NP. This is summarized in the following theorem:

Theorem 1 *The OAC, OTELC, and OMELC problems are NP-complete.*

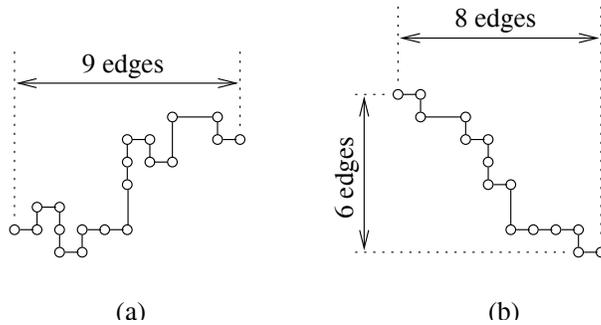


Figure 3: (a) a path, monotonic with respect to the x axis, with $\ell_x \geq 9$, and (b) a path, monotonic with respect to both the x axis and the y axis, with $\ell_x \geq 8$ and $\ell_y \geq 6$.

3 Tools and Basic Gadgets

In this section we introduce a class of orthogonal representations admitting an exponential number of orthogonal grid drawings with minimum area. In each orthogonal grid drawing with the minimum area of such orthogonal representations, the basic blocks that compose the graph are themselves drawn with minimum area, so that the property of being drawn with minimum area is somehow “inherited” by these subgraphs from the whole drawing.

This is imposed by carefully exploiting the properties of suitably folded paths. Such paths can be used to “fill” differently shaped void spaces, i.e., to cover all the grid points of a void space (whatever its shape is) in each orthogonal grid drawing with minimum area of the orthogonal representation of the whole graph.

A more formal discussion of the above topics will be given in the next two subsections. In what follows we introduce some definitions and properties.

An orthogonal representation of a graph consisting of a single path p can be more easily described by specifying an end vertex v of p and the sequence of angle labels encountered when traversing the external face from v to the other end vertex of p . In what follows we will call *turn sequence* the sequence of angle labels of p , and will denote by x^n the succession of n occurrences of label x in a turn sequence.

An orthogonal representation of a path p is *monotonic* with respect to the x (y) axis if, in each orthogonal grid drawing of p , the x (y) coordinates of the vertices of p are decreasing, increasing, non decreasing, or non increasing, when p is traversed from one end vertex to the other.

Let Γ be an orthogonal grid drawing of an orthogonal representation of a path p monotonic with respect to the x (y) axis. The following two properties hold:

Property 1 *Let k be the number of edges of p parallel to the x (y) axis, we have that $\ell_x^\Gamma \geq k - 1$ ($\ell_y^\Gamma \geq k - 1$).*

Property 2 *If $\ell_x^\Gamma = \ell_x$ ($\ell_y^\Gamma = \ell_y$), the horizontal (vertical) distance between any two vertices of p has the same value in Γ as in any other orthogonal grid drawing Γ' with $\ell_x^{\Gamma'} = \ell_x$ ($\ell_y^{\Gamma'} = \ell_y$).*

Some examples of Properties 1 and 2 are given in Figure 3.

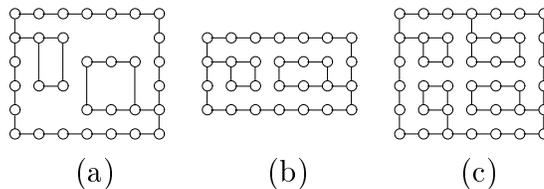


Figure 5: While the orthogonal grid drawing in (a) occupies the minimum area, its subgraphs are not themselves drawn with minimum area. Conversely, in (b) and (c), the property of being drawn with minimum area is both a global and a local property.

type t_2 or t_4 vertical edge can not be on the same vertical grid line with another vertical edge of type t_2 or t_4 . It follows that in each orthogonal grid drawing of the path p that uses at most four horizontal grid lines, each vertical grid line can host at most a single vertical edge of the path p , and, consequently, the proposed orthogonal grid drawing of the path p minimizes the number of vertical grid lines too.

Regarding the uniqueness of the proposed solution, consider the sequence of edge types encountered when traversing p . Observe that such sequence is a repetition of the subsequence: t_1, t_2, t_3, t_4 . Let R be a partial order relation between two vertical edges of p . Namely, given two vertical edges e and e' of p , we say that $e <_R e'$ if the x coordinate of e is smaller than the x coordinate of e' in any orthogonal grid drawing of p . Observe that, if e' is the vertical edge immediately following the vertical edge e in the path p , then $e <_R e'$ or $e' <_R e$, depending on the type of e (see Fig. 4.c, 4.d, 4.e, and 4.f). From the fact that a solution of the given path compaction problem is an orthogonal grid drawing Γ of p such that each vertical grid line hosts a single vertical edge, it follows that a solution of the given path compaction problem corresponds to a linear order relation $R' \supset R$. The proof is completed by showing that any linear order relation, different from the one implicit in the drawing of Fig. 4, corresponds to an orthogonal grid drawing of p with an area greater than the one used by the proposed solution. \square

3.2 Inheriting a not Hereditary Property

The property of being drawn with minimum area is a global property, regarding the whole drawing, and does not necessarily reflect on parts of it. Fig. 5.a provides an example in which the area covered by the external box is minimum, while the subgraphs contained inside are not themselves drawn as small as they could be.

Obviously, we can adjust the size of the external box so to force the global optimality to imply a local optimality, as in Fig. 5.b, or add a suitable number of objects to the drawing to obtain the same effect (Fig. 5.c), but in doing this we limit the number of optimal solutions, i.e., we tend to make the inside objects immovable and the orthogonal drawing of the whole graph fixed.

What we need is a systematic way to make the area optimality an inherited property while preserving a suitable degree of “freedom” (i.e., number of alternatives) for the orthogonal grid drawing of the whole graph.

Here we will tackle a particular instance of this problem, that we call (n, w, h, g) -*sliding rectangles compaction* problem, in which, the inside subgraphs can be modeled by n contiguous (w, h) -compactable rectangles. Each rectangle can slide vertically with respect to the following one of at least g grid lines. Simply by inserting further subgraphs

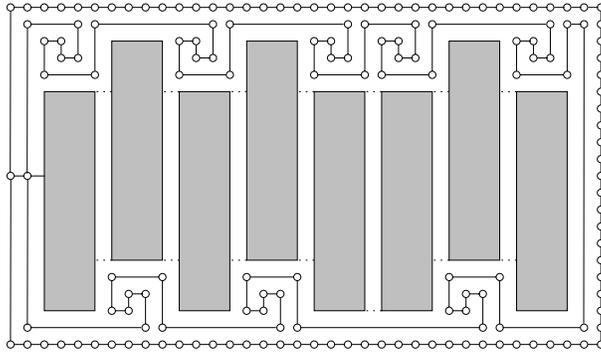


Figure 6: The $(n, 3, h, 3)$ -sliding rectangles press gadget, in the particular case of $n = 8$ and $h = 14$. The figure shows an orthogonal grid drawing with minimum area in which the 2nd, 4th, and 7th rectangles are drawn 3 grid lines over the others.

and choosing the number of vertices of the sides of the external box, we would like to obtain an orthogonal representation that admits an exponential number of orthogonal grid drawings with minimum area in all of which the rectangles inside inherit the same property (i.e., are themselves drawn with minimum area).

As a solution for the $(n, 3, h, 3)$ -sliding rectangles compaction problem we produce the $(n, 3, h, 3)$ -sliding rectangles press (see Fig. 6) composed as follows: the box around the graph has top and bottom side $4n + 4$ vertices long and right side $h + 7$ vertices long; the rectangles are circled with a belt consisting of a path with turn sequence $\sigma = (r^4 l^4)^n r^4$; the first rectangle, the belt, and the external box are linked together as shown in Fig. 6.

Lemma 1 *The $(n, 3, h, 3)$ -sliding rectangles press gadget admits an exponential number of orthogonal grid drawings with the minimum area $(4n + 3) \times (h + 6)$. In each of such orthogonal grid drawing the rectangles are necessarily drawn themselves with the minimum area $3 \times h$.*

Proof: Property 1 applied to the top and right sides of the external boundary gives $\ell_x \geq 4n + 3$ and $\ell_y \geq h + 6$. Suppose all sliding rectangles are positioned as shown in Fig. 7.a. The turn sequence $\sigma = (r^4 l^4)^n r^4$ of the belt can be rewritten $r(r^3 l^4 r)^n r^3$, reflecting the fact that the first r label and the last r^3 labels are used to surround the rectangles. Observe that above the rectangles a space of 4 horizontal grid lines and $4n$ vertical grid lines is left. For Property 4 such a space is exactly the space needed to host the remaining sub-sequence $(r^3 l^4 r)^n$. It follows that exists an orthogonal grid drawing of the $(n, 3, h, 3)$ -sliding rectangles press gadget in the minimum area $(4n + 3) \times (h + 6)$ in which all the rectangles are themselves drawn with minimum area $3 \times h$.

Suppose a sliding rectangle is moved up of 4 horizontal grid lines. The darkened area of Fig. 7.b is covered by the rectangle, splitting the above space in two, while the darkened area of Fig. 7.c, is freed at its bottom, to constitute a third space. From the consideration that removing a sub-sequence $r^3 l^4 r$ (black vertices of Fig. 7.b) from the turn sequence σ , and adding a sub-sequence $r l^4 r^3$ before the trailing r label (black vertices of Fig. 7.c), yields $r(r^3 l^4 r)^{n-1} r^2 (r l^4 r^3) r = r(r^3 l^4 r)^{n-1} (r^3 l^4 r) r^3 = r(r^3 l^4 r)^n r^3 = (r^4 l^4)^n r^4 = \sigma$, follows that the same turn sequence σ can be adjusted in the three spaces (two above the rectangles and one below them). Furthermore, due to Property 4, a sliding rectangle can not have an intermediate position, and has $\ell_x^\Gamma = 3$ and $\ell_x^\Gamma = h$ in any orthogonal grid

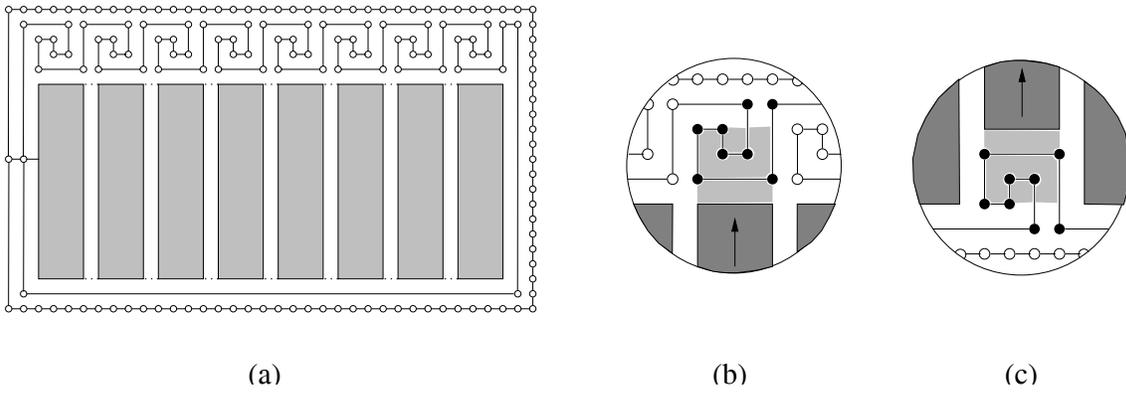


Figure 7: (a) An $(n, 3, h, 3)$ -sliding rectangles press gadget with all rectangles in the same lower position (used in the proof of Lemma 1); when a rectangle slides from the lower to the upper position the darkened area in (b) is covered and the darkened area in (c) is left behind; black vertices can be removed in (b) and can be inserted in (c).

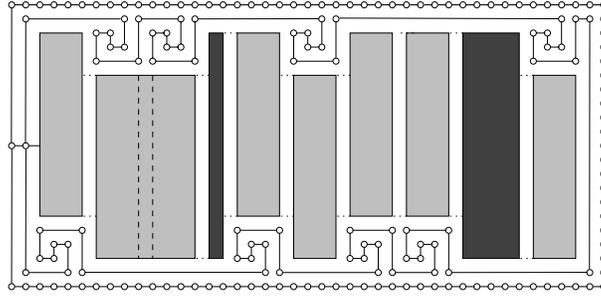


Figure 8: A variant of the $(n, 3, h, 3)$ -sliding rectangles press gadget obtained by inserting two immovable rectangles (Property 5) and replacing a $(3, h)$ -compactable sliding rectangle with a $(7, h)$ -compactable one (Property 6).

drawing Γ . Analogous considerations can be used to prove that any other configuration of the rectangles in the upper or lower position corresponds to an orthogonal grid drawing with minimum area, in which all sliding rectangles are themselves drawn with minimum area. \square

The following properties introduce some variants to the sliding rectangles press gadget, for which an accordingly modified version of Lemma 1 holds.

Property 5 *In the $(n, 3, h, 3)$ -sliding rectangles press gadget, a $(w', h + 3)$ -compactable rectangle, with w' arbitrary, can be inserted at any position between the sliding rectangles, provided that $w' + 1$ vertices are added to the top and bottom sides of the external box.*

Property 6 *In the $(n, 3, h, 3)$ -sliding rectangles press gadget, a $(3, h)$ -compactable rectangle can be replaced by a $(3 + 4c, h)$ -compactable one, where c is an arbitrary positive integer, provided that $4c$ vertices are added to the top and bottom sides of the external box, and a $(r^{4l^4})^c$ sub-sequence is inserted at the beginning of the turn sequence of the belt.*

Fig. 8 shows a sliding rectangles press gadget featuring both variants.

Property 7 For an $(n, 3, h, 3)$ -sliding rectangles press gadget such that the sliding rectangles can assume only a subset of all the possible (otherwise exponential) configurations, an accordingly modified version of Lemma 1 holds, stating that only the corresponding subset of orthogonal grid drawings with minimum area are admitted.

4 NP-hardness of the OAC Problem

We prove that the Orthogonal Area Compaction problem is NP-hard by means of a reduction from the SAT problem:

Problem: Satisfiability (SAT)

Instance: A set of clauses, each containing literals from a set of boolean variables.

Question: Can truth values be assigned to the variables so that each clause contains at least one true literal?

Given a formula ϕ in conjunctive normal form with variables x_1, \dots, x_n and clauses C_1, \dots, C_m , we produce an orthogonal representation $H_A(\phi)$ and a constant $K_A(\phi)$ such that an orthogonal grid drawing of area less or equal than $K_A(\phi)$ exists if and only if ϕ is satisfiable.

Please, notice that in the SAT definition all the variables in the same clause can be assumed to be different, i.e., the version of SAT in which each clause contains appearances of *distinct* variables is also NP-complete (this can be trivially proved by introducing a linear number of dummy variables and further clauses).

In what follows we show how to build the instance (H_A, K_A) of the OAC problem corresponding to an instance ϕ of the SAT problem (subsection 4.1), and prove that a solution to the OAC problem on instance (H_A, K_A) exists if and only if the corresponding instance ϕ of the SAT problem is satisfiable (subsection 4.2).

4.1 Instance (H_A, K_A) Construction Rules

In this subsection we describe how an instance ϕ of the SAT problem maps to an instance $(H_A(\phi), K_A(\phi))$ of the OAC problem.

The construction of the orthogonal representation $H_A(\phi)$ requires three steps:

- i) build a clause-gadget for each clause C_i ,
- ii) combine clause-gadgets together and add hinges, and
- iii) add external boundary and belt.

These three steps, and the mentioned subgraphs, are described in the following three paragraphs. The last paragraph is concerned with producing a value for $K_A(\phi)$

i) Clause-gadget Construction

In the following we assume that the formula ϕ of the SAT problem has n boolean variables, x_1, \dots, x_n , and m clauses C_1, \dots, C_m .

The clause-gadget is composed by n *chambers*, one for each variable, whether the variable actually occurs in the clause C_i or not. We call (i, j) -chamber the chamber of

clause C_i corresponding to the variable x_j . The (i, j) -chamber, with $1 < j < n$ is shown in Fig. 9.a, while the $(i, 1)$ -chamber and the (i, n) -chamber are shown in Fig. 9.b and 9.c, respectively.

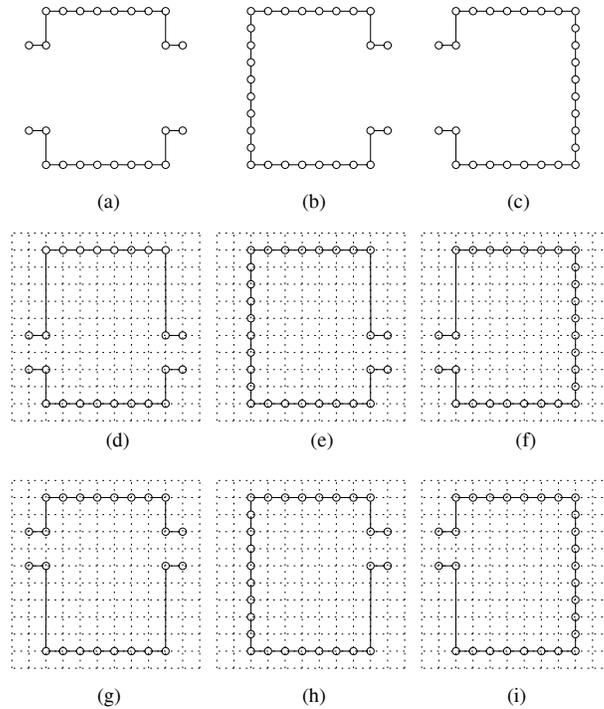


Figure 9: First line: the chambers corresponding to: (a) a variable x_j , with $1 < j < n$; (b) the variable x_1 ; and (c) the variable x_n . Second line: true-compliant orthogonal grid drawings corresponding to the orthogonal representations of figure (a), (b), and (c), respectively. Third line: false-compliant orthogonal grid drawings corresponding to the same orthogonal representations.

Observe that the edge lengths of Fig. 9.a, 9.b, and 9.c are not meaningful, since such figures are only meant to describe orthogonal representations. However some peculiar orthogonal grid drawings of the chambers will be so recurrent in what follows to deserve a definition: we define *true-compliant* an orthogonal grid drawing of a chamber such that the vertex distances are exactly those represented in Fig. 9.d, 9.e, or 9.f, and *false-compliant* an orthogonal grid drawing of a chamber such that the vertex distances are exactly those represented in Fig. 9.g, 9.h, or 9.i.

For the sake of brevity we call *compliant* a true-compliant or false-compliant orthogonal grid drawing of a chamber, and we say that, in a given orthogonal grid drawing Γ of H_A , a chamber is true-compliant (false-compliant, compliant, respectively) whenever the orthogonal grid drawing of the chamber induced by Γ is true-compliant (false-compliant, compliant, respectively).

All the n chambers corresponding to clause C_i are attached together in a row, in such a way that the (i, j) -chamber shares two vertices with the $(i, j + 1)$ -chamber. We call such vertices *weldings*. Fig. 10 shows all the chambers of a clause-gadget for a formula with six variables.

To be completed the clause-gadget is added with two types of further subgraphs: obstacles, and pathways. An (i, j) -chamber corresponding to a variable x_j not occurring

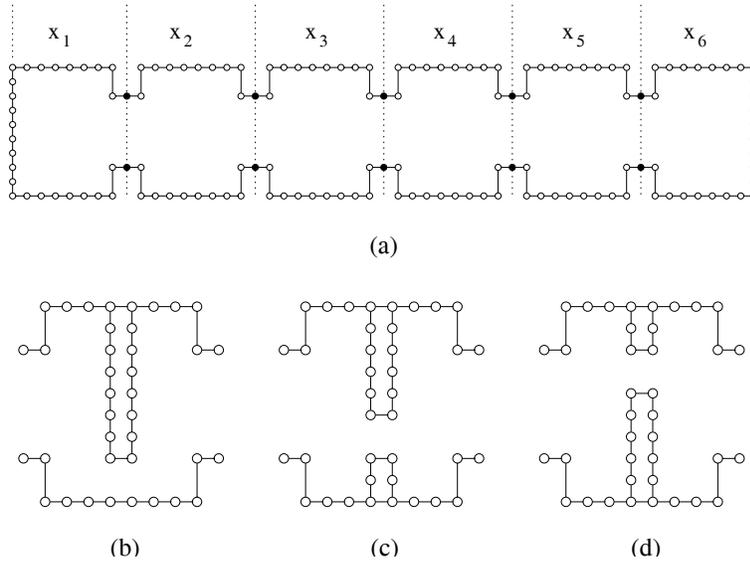


Figure 10: (a) The chambers corresponding to a clause of a formula with six variables. The black vertices are weldings. The obstacles inserted in a (i, j) -chamber, when variable x_j does not occur in clause C_i (b); when variable x_j occurs in the clause C_i with a positive literal (c); or a negative literal (d). The figure shows only the case of an (i, j) -chamber with $1 < j < n$; other cases are similar.

in the clause C_i receives an *obstacle* as shown in Fig. 10.b. Any other (i, j) -chamber receives two obstacles as shown in Fig. 10.c, if the variable x_j occurs with a positive literal, or as shown in Fig. 10.d, otherwise. Fig. 11.a shows an example of a clause-gadget with its obstacles.

Finally, the clause-gadget is added with a *pathway* composed by a succession of $2n - 1$ A-shaped structures linked together as shown in Fig. 11.b. The pathway originates from the $(i, 1)$ -chamber and terminates in the (i, n) -chamber, as shown in the same figure.

ii) Combining Clause-gadgets Together

All the clause-gadgets corresponding to formula ϕ are placed one upon the other, so that each (i, j) -chamber shares its bottom 8 vertices with the $(i + 1, j)$ -chamber, for $i = 1, \dots, n - 1$. Furthermore, hinges are introduced. *Hinges* are vertical paths, originating from the weldings.

A hinge 8 vertices long links the welding between the (i, j) -chamber and the $(i, j + 1)$ -chamber with the welding between the $(i + 1, j)$ -chamber and the $(i + 1, j + 1)$ -chamber, for $i = 1, \dots, m - i$, and $j = 1, \dots, n - 1$.

A hinge 6 vertices long attaches to the welding between the (i, j) -chamber and the $(i, j + 1)$ -chamber, with $i = 1$ or $i = n$, and $j = 1, \dots, n - 1$. The clause-gadgets and hinges for a formula with five variables and four clauses are shown in Fig. 12.a.

iii) Adding External Boundary and Belt

To obtain the final orthogonal representation $H_A(\phi)$ an external boundary and a belt are added to the construction. The *external boundary* has a top and bottom sides of $9n + 3$ vertices and a right side of $9m + 8$ vertices. The *belt* is a path inserted between the

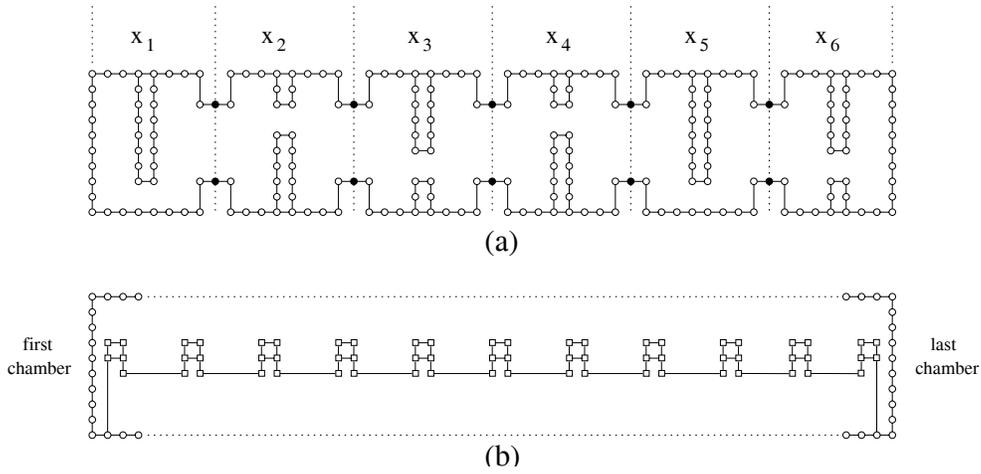


Figure 11: (a) The chambers corresponding to clause $C_i = \bar{x}_2 \vee x_3 \vee \bar{x}_4 \vee x_6$ of a formula ϕ with six variables once completed with obstacles. The black vertices are weldings. (b) The pathway inserted in the clause-gadget is composed by 11 (i.e., $2n - 1$) A-shaped structures linked together.

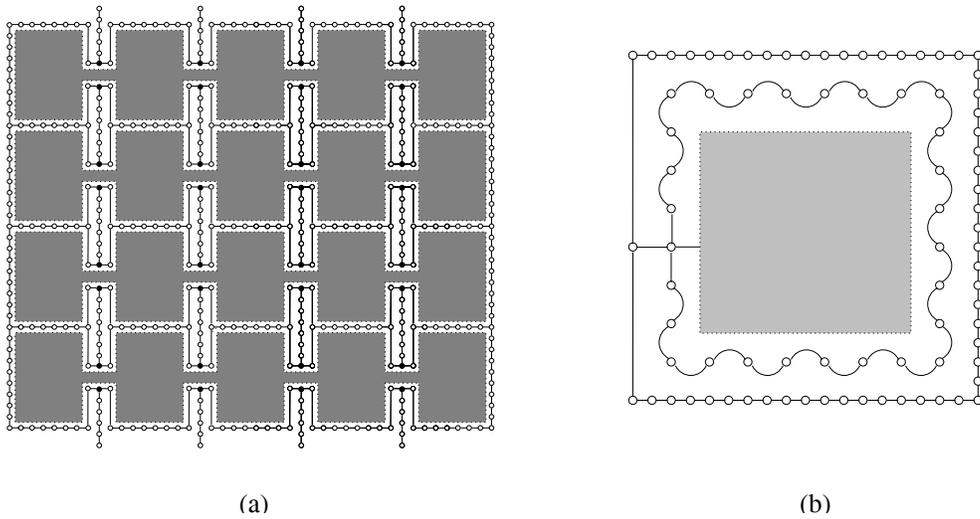


Figure 12: (a) Clause-gadgets and hinges for a formula with five variables and four clauses. The black vertices are weldings. The inside of the clause-gadgets is not represented (darkened areas). (b) Adding external boundary and belt to the construction of (a) (darkened area) to obtain the final orthogonal representation $H_A(\phi)$.

boundary and the core of the construction and composed by $2 + 24(n - 1)$ vertices, so that its turn pattern is $(r^{4l^4})^{2nr^4}$. The external boundary, the belt, and the core of the construction are attached together as shown in Fig. 12.b.

iv) Computing Constant $K_A(\phi)$

The instance (H_A, K_A) of the OAC problem is completely defined as the value of $K_A(\phi) = (9n + 2) \times (9m + 7)$ is assigned. Figures 13 and 14 show two examples of orthogonal grid drawings of $H_A(\phi)$ for a formula ϕ with four boolean variables and four clauses with $K_A(\phi) = 38 \times 43 = 1,634$.

4.2 Correctedness

Here we prove that an orthogonal grid drawing Γ of area at most $K_A(\phi)$ can be found for the orthogonal representation $H_A(\phi)$ if and only if the corresponding instance ϕ of the SAT problem admits a solution.

Property 8 *For the orthogonal representation $H_A(\phi)$ we have that $\ell_x \geq 9n + 4$ and $\ell_y \geq 9m + 7$.*

Proof: It suffices applying Property 1 to the top side and right side of the external boundary of the orthogonal representation $H_A(\phi)$ (see Fig. 12.b). \square

As a consequence of the previous property we have that the minimum area of an orthogonal grid drawing of $H_A(\phi)$ can not be less than $K_A(\phi)$, being $K_A(\phi)$ by definition equal to $(9m + 2) \times (9m + 7)$, and that for any orthogonal grid drawing Γ of area $K_A(\phi)$ is $\ell_x^\Gamma = 9m + 2$ and $\ell_y^\Gamma = 9m + 7$.

Let Γ be an orthogonal grid drawing of $H_A(\phi)$ with minimum area. The following property hold:

Property 9 *The horizontal (vertical) distance between a welding v and a vertex on a vertical (horizontal) side of the external boundary has the same value in Γ as in any other orthogonal grid drawing Γ' of $H_A(\phi)$ with minimum area.*

Proof: For the horizontal distance between v and a vertex on a vertical side of the external boundary it suffices considering Property 2. Regarding the vertical distance between a welding and the external boundary, notice that, when the area of the orthogonal grid drawing is minimum, all the welding between (i, j) -chamber and $(i, j + 1)$ -chamber, with $i = 1, \dots, n$ have the same x coordinate, i.e., the hinges that attach to them lay necessarily on the same vertical grid line. Considering that hinges have $\ell_y \geq 8$ or $\ell_y \geq 6$, depending on the number of their vertices, and that at least a horizontal grid line must be left between two hinges, to host the pathway, and between the external boundary and the hinges, to host the belt, the statement follows. \square

Property 10 *For an (i, j) -chamber we have that $\ell_x = 9$ and $\ell_y = 9$.*

Proof: For the width it suffices applying Property 1 to the top sides of the subgraphs represented in Fig. 9.a, 9.b, and 9.c.

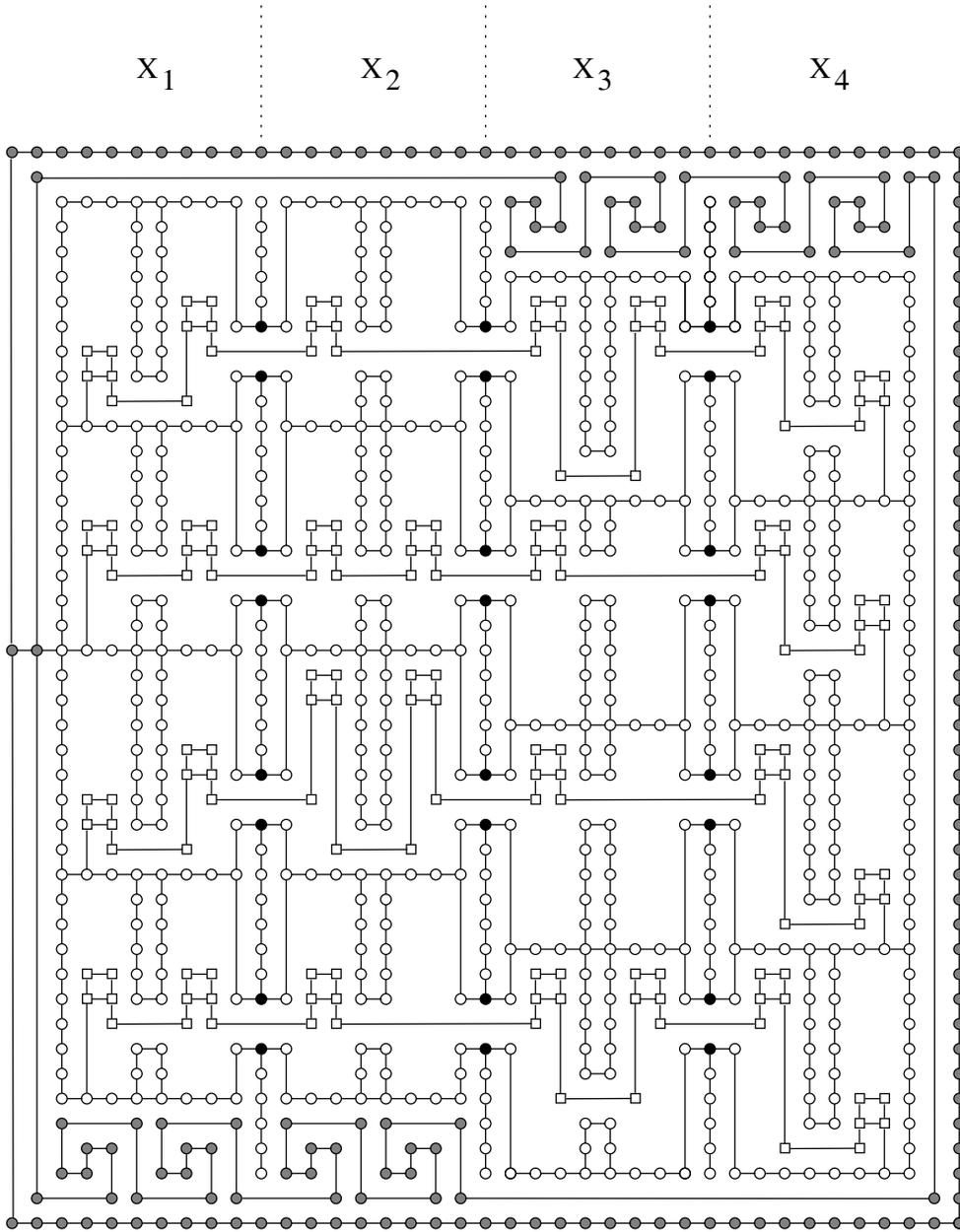


Figure 13: The orthogonal representation $H_A(\phi)$ corresponding to formula $\phi = (x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$. The orthogonal grid drawing corresponds to the truth assignment: $x_1 = true, x_2 = true, x_3 = false, x_4 = false$.

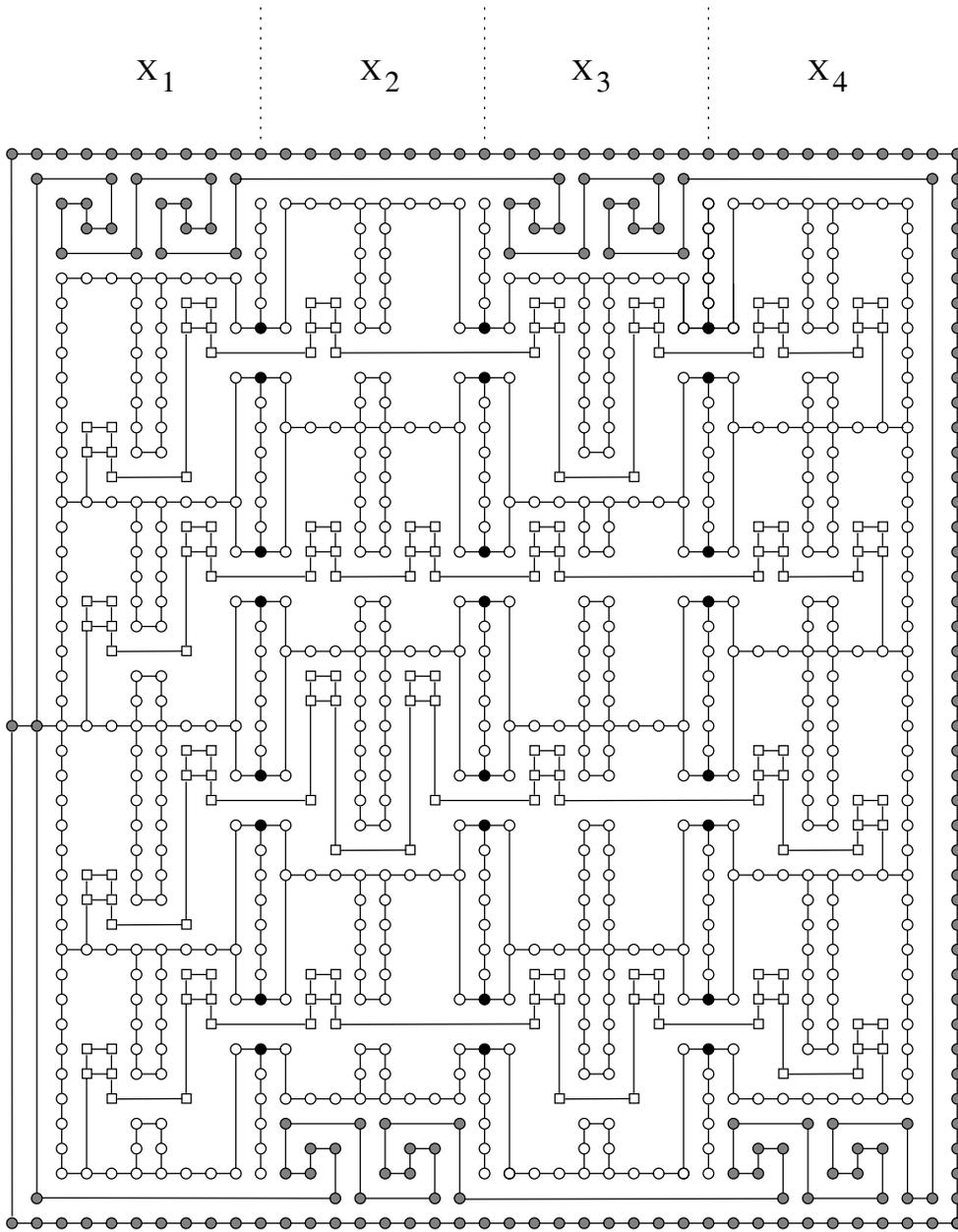


Figure 14: The orthogonal representation $H_A(\phi)$ corresponding to the same formula of Figure 13 $\phi = (x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$. The orthogonal grid drawing corresponds to the truth assignment: $x_1 = false, x_2 = true, x_3 = false, x_4 = true$.

For the height the proof is trivial when $j = 1$ or $j = n$ (consider applying Property 1 on the left and right side of Fig. 9.b, and 9.c, respectively). For other values of j , observe that the chamber is composed by two non connected subgraphs, and that at least a line must lay between them to host the pathway. \square

Property 11 *A compliant (i, j) -chamber corresponding to a variable x_j not occurring in clause C_i contains two A-shaped structures of the pathway.*

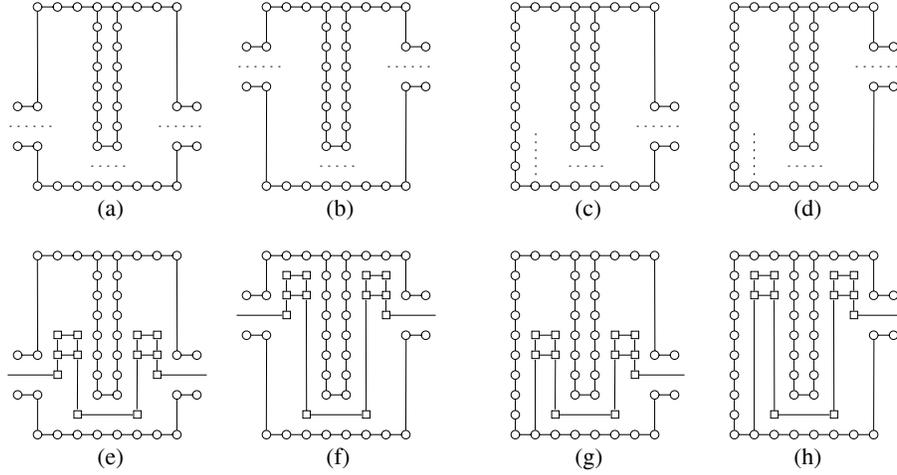


Figure 15: A compliant (i, j) -chamber corresponding to a variable x_j not occurring in the clause C_i contains two A-shaped structures of the pathway. The first line shows the possible configurations. The second line shows the only possible solutions. Two A-shaped structures are always involved.

Proof: The statement follows trivially from the observation that the pathway must overlap with the dotted lines shown in Fig. 15. From the same figure is apparent how the only way to accomplish this is by inserting two A-shaped structures of the pathway inside an (i, j) -chamber. Fig. 15 shows that two A-shaped structures of the pathway are contained both in a true-compliant and in a false-compliant (i, j) -chamber, for $i = 1, \dots, n$. \square

Property 12 *A true-compliant (false-compliant) (i, j) -chamber contains two A-shaped structures of the pathway if the corresponding literal is negative (positive). May contain only one if the literal is positive (negative).*

Proof: The proof is obvious considering Fig. 16 and Fig. 17 where the cases are represented. \square

Let Γ be an orthogonal grid drawing of H_A . We say that a clause-gadget C_i is *compliant* in Γ if each chamber of C_i is compliant in Γ . Also, we define *truth configuration* of C_i in Γ the succession of boolean values b_j , $j = 1, \dots, n$, such that b_j is true (false) if the corresponding (i, j) -chamber is true-compliant (false-compliant).

Lemma 2 *A clause-gadget admits a truth configuration T if and only if assigning the sequence of boolean values of T to the variables x_1, \dots, x_n produces at least one true literal in the corresponding clause C_i .*

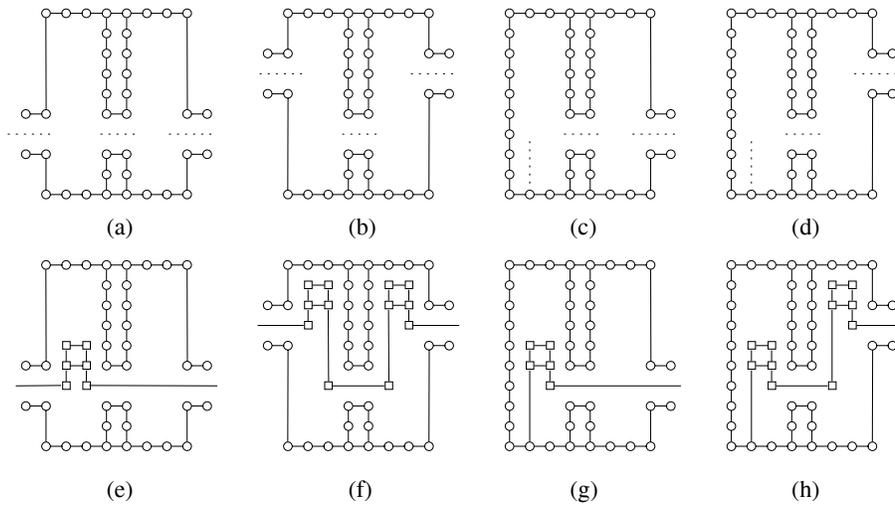


Figure 16: When variable x_j occurs in the clause C_i with a positive literal, a true-compliant (i, j) -chamber (a and c), may contain only one A-shaped structure of the pathway (e and g, respectively), while a false-compliant (i, j) -chamber (b and d) contains necessarily two A-shaped structures (f and g, respectively).

Proof: By contradiction: suppose a clause-gadget admits a truth configuration T , and that all literals of the corresponding clause C_i yield a false value when the sequence of boolean values of T is assigned to the variables x_1, \dots, x_n . Since all chambers of the clause-gadget correspond to variables not occurring in C_i , or occurring with an opposite truth value, each chamber must contain two A-shaped structures of the pathway (Properties 11 and 12, respectively). It follows that the number of A-shaped structures of the pathway of the clause-gadget is $2n$, while, by construction, it is $2n - 1$.

Conversely, suppose T is a truth configuration that produces at least a true literal (say for the x_t variable) in the clause C_i when assigned to the variables x_1, \dots, x_n . For Property 12 the (i, t) -chamber admits a compliant orthogonal grid drawing containing only one A-shaped structure of the pathway. Since $2n - 2$ A-shaped structures and $n - 1$ chambers are left, each chamber contains 2 of them, and can be compliant according to the corresponding truth value of the truth configuration T (Properties 11, and 12). \square

Lemma 3 *An orthogonal grid drawing of area at most $K_A(\phi)$ of the orthogonal representation $H_A(\phi)$ exists if and only if formula ϕ is satisfiable.*

Proof: Suppose formula ϕ is satisfiable: Lemma 2 implies that an orthogonal grid drawing exists such that all clause-gadgets assume the same truth configuration. It follows that each vertical column of chambers corresponding to the same boolean variable covers a rectangular area of $7 \times 9m$. The orthogonal representation $H_A(\phi)$ is a $(n, 3, 9m, 4)$ -sliding rectangles press gadget. In fact:

- Each rectangle is $(3 + 4c, 9m)$ -compactable, with $c = 1$, as allowed by Property 6.
- Between each pair of contiguous sliding rectangles a $(0, 9m + 4)$ -rectangle is inserted, as allowed by Property 5.
- Furthermore, according to the above two variants to the $(n, 3, 9m, 4)$ -sliding rectangles press gadget:

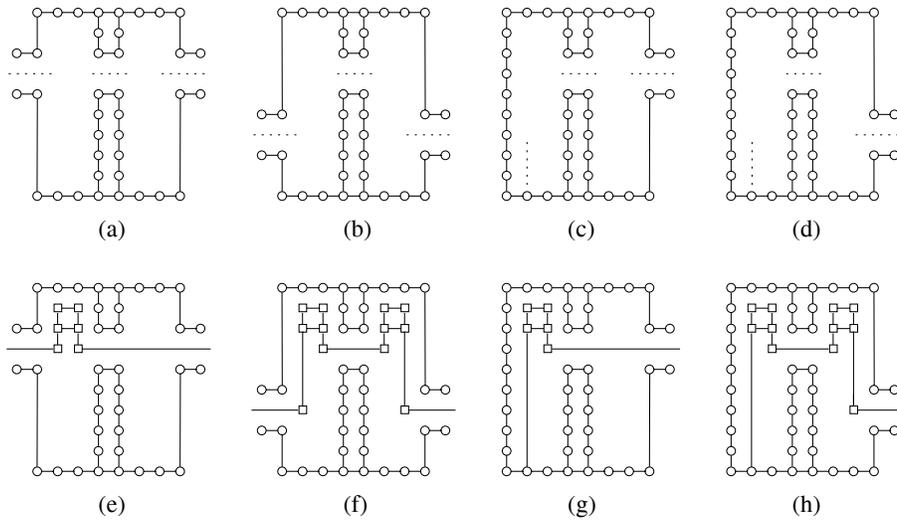


Figure 17: When variable x_j occurs in the clause C_i with a negative literal, a false-compliant (i, j) -chamber (a and c), may contain only one A-shaped structure of the pathway (e and g, respectively), while a true-compliant (i, j) -chamber (b and d) contains necessarily two A-shaped structures (f and g, respectively).

- the top and bottom sides of the external boundary are $9n + 3$ vertices long,
- the right side of the boundary is $9m + 8$ vertices long, and
- the belt has a bend pattern $\sigma = (r^4 l^4)^{2n} r^4$.

Lemma 1 and Property 7 assure that an orthogonal grid drawing with area $K_A(\phi)$ exists.

Conversely, suppose formula ϕ is not satisfiable. Lemma 2 implies that there isn't a truth configuration that can be assumed by all clause-gadgets. Since each chamber is attached to the chamber below with its bottom side vertices, from the previous statement follows that in any orthogonal grid drawing of $H_A(\phi)$, at least one clause-gadget is not compliant, i.e., at least one of its chambers is not compliant. As a consequence in any orthogonal grid drawing of $H_A(\phi)$ one of the following holds:

1. a clause-gadget has length greater than $9n$,
2. all clause-gadget have length equal to $9n$, and a column of hinges has height greater than $9m + 4$, or
3. none of the previous conditions hold, and at least one column of chambers has height greater than $9m$.

The proof is completed by showing that each of the above three statements implies that the orthogonal grid drawing Γ of $H_A(\phi)$ has an area greater than $K_A(\phi)$. In fact, case 1 implies $\ell_x^\Gamma > \ell_x$, and from Property 8 and the definition of $K_A(\phi)$, the statement follows; case 2 implies analogously that $\ell_y^\Gamma > \ell_y$; and for case 3 the statement follows trivially from the fact that Lemma 1 rules out the existence of an orthogonal grid drawing of the $(n, 3, 9m, 4)$ -sliding rectangles press gadget with area $K_A(\phi)$ in which a rectangle has an area greater than $7 \times 9m$. \square

Lemma 4 *The OAC problem is NP-hard.*

Proof: The statement follows from Lemma 3 and from the fact that the orthogonal representation $H_A(\phi)$ has $O(n \times m)$ vertices, and its construction (and the computation of $K_A(\phi)$) can be done in polynomial time. \square

5 NP-Hardness of the OTELC and OMELC Problems

To prove that the Orthogonal Total Edge Length Compaction problem is NP-hard we reduce the SAT problem to it, slightly modifying the construction described in Section 4.

Observe that, in any orthogonal grid drawing of H_A with area K_A , the total edge length can not be greater than $l_0 = (w + 1) \times (h + 1)$, where h and w are the minimum values of ℓ_y and ℓ_x for H_A , respectively. To obtain H_{TEL} we add to H_A a number of l_0 edges along the top and right sides of H_A and connect them to H_A as shown in Fig. 18.a.

We assign to K_{TEL} the value $l_0(w + 2) + l_0(h + 2) + l_0 = l_0(w + h + 5)$, so that when H_A has an orthogonal grid drawing with area K_A , H_{TEL} has an orthogonal grid drawing with total edge length less or equal than K_{TEL} . Conversely, if H_{TEL} has an orthogonal grid drawing of total edge length less or equal than K_{TEL} , then H_A has an orthogonal grid drawing with area K_A . In fact, it's easy to see that every orthogonal grid drawing Γ of H_{TEL} such that H_A covers an area bigger than K_A has a total edge length greater than K_{TEL} .

Then, from Lemma 3 follows that the corresponding formula ϕ has a solution.

Similarly, the SAT problem can be reduced to the Orthogonal Maximum Edge Length Compaction problem. Namely, to obtain the instance (H_{MEL}, K_{MEL}) of the OMELC problem we modify the orthogonal representation H_A , adding a rectangular box to it, in such a way that the number of vertices along the top side of the obtained orthogonal representation is equal to the number of the vertices along the right side of it (see Fig. 18.b and 18.c). Finally, we add a pair of edges running along the top and right side of the construction as shown in the same figures.

Since the last added two edges are the longest in any orthogonal grid drawing of H_{MEL} , an orthogonal grid drawing of H_{MEL} that minimizes the maximum length, also minimizes the perimeter of H_A . In particular, when the maximum edge length of H_{MEL} is $K_{MEL} = \max(9n + 2, 9m + 7)$, the perimeter of H_A is exactly $(9n + 2) \times (9m + 7)$, and the area of H_A is exactly K_A , so that, an orthogonal grid drawing of H_{MEL} with maximum edge length equal to K_{MEL} exists if and only if the corresponding formula ϕ has a solution.

The following lemma is then proved:

Lemma 5 *The OTELC, and OMELC problems are NP-hard.*

6 The OAC, OTELC, and OMELC Problems are in NP

To prove that the three problems are in NP we produce three nondeterministic Turing machines that decide them in polynomial time.

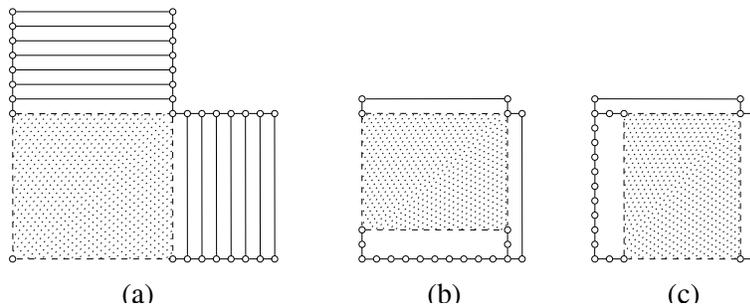


Figure 18: (a) The orthogonal representation $H_{TEL}(\phi)$, and (b and c) the two possible cases for the orthogonal representation $H_{MEL}(\phi)$. The darkened areas represent the $H_A(\phi)$ orthogonal representation of Fig. 12.b.

Such nondeterministic Turing machines take as input the instance (H, K) , and generate the set S of orthogonal grid drawings of H with coordinates in the range $[0, v - 1]$, where v is the number of vertices of H . Finally, each orthogonal grid drawing in S is checked to verify in polynomial time if its area, total edge length, or maximum edge length, respectively, is less or equal than the constant K .

It's easy to show that, if an orthogonal grid drawing $\Gamma \notin S$ of the orthogonal representation H exists with area (total edge length, maximum edge length, respectively) less or equal than the constant K , then an orthogonal grid drawing $\Gamma' \in S$ exists with equal or less area (total edge length, maximum edge length, respectively). In fact, since our orthogonal representations have no bends, if Γ is an orthogonal grid drawing of H such that the horizontal (vertical) distance between two of its vertices is bigger than $v - 1$, then Γ necessarily contains a vertical (horizontal) grid line non intersecting any vertex of H that can be removed, decreasing the distance of such two vertices, the area, the total edge length and, possibly, the maximum edge length. Furthermore, if $\Gamma \notin S$ is an orthogonal grid drawing of H such that the distance of any two vertices is less or equal to $v - 1$, then an orthogonal grid drawing $\Gamma' \in S$ exists with the same area (total edge length, maximum edge length, respectively).

Observe that, being the coordinates of the vertices in the range $[0, v - 1]$, it's easy to check in polynomial time whether an orthogonal grid drawing $\Gamma \in S$ is a feasible solution (vertices do not overlap, edges are orthogonal and do not intersect, angles around vertices are coherent with H labeling), and whether the area, total edge length, or maximum edge length of Γ is less or equal than the constant K .

The nondeterministic Turing machine for the OAC problem works as follows: it takes as input the instance (H, K) , and, if v is the number of vertices of H , writes an arbitrary sequence of v coordinate pairs in the range $[0, v - 1]$. When this writing stops, the machine goes back and checks to see whether the string written is an orthogonal grid drawing, and, if so, whether its area is less or equal than K . Similar nondeterministic Turing machines can be easily devised for the OTELC and OMELC problems. The following lemma is, therefore, proved:

Lemma 6 *The OAC, OTELC, and OMELC problems are in NP.*

7 Approximability Considerations

As usual, once NP-completeness has been established for an optimization problem, the question about its approximability arises. It's easy to prove that the three optimization problems related with the OAC, OTELC, and OMELC problem, respectively, are not in PTAS, i.e., that they don't allow a polynomial-time approximation scheme. For the area minimization problem, a proof may consist in attaching together a $q \times q$ array of $H_A(\phi)$ orthogonal representations, as shown in Figure 19, and considering that any orthogonal grid drawing of the whole construction with area "sufficiently near" to the optimal value would have at least one $H_A(\phi)$ drawn with minimum area. Similar constructions can be devised to prove that the total and maximum edge length minimization problems are not in PTAS as well.

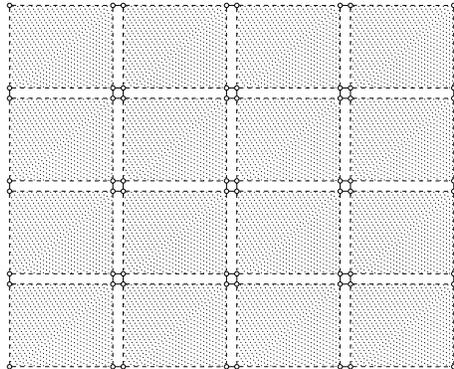


Figure 19: An array of 4×4 $H_A(\phi)$ orthogonal representations that can be used to prove that the problem of minimizing the area of an orthogonal grid drawing does not allow a polynomial-time approximation scheme.

8 Conclusions and Open Problems

In this paper we have shown that compacting an orthogonal representation while minimizing an aesthetic measure between area, maximum edge length, and total edge length is an NP-complete problem and that it doesn't allow a polynomial-time approximation scheme.

An interesting topic is whether the three problems retain their complexity when focusing on peculiar classes of graphs. One may ask, for example, what is the influence of the connectivity properties of the graphs. For biconnected graphs, in spite of the fact that the proposed constructions are not biconnected (due to hinges and belt attachment), it's easy to modify these parts (thickening them as shown in Fig. 20) so to produce an orthogonal representation $H_A(\phi)$ of a biconnected graph.

Other interesting problems are the following: does an orthogonal representation, whose underlying graph is a simple cycle, retain the complexity of the three general problems? Does "turn-regularity" (defined in [2]) characterize the orthogonal representations for which the compaction problem is polynomial?

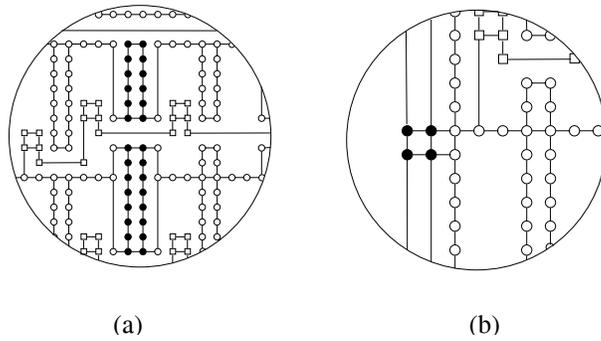


Figure 20: Hinges and belt attachment can be thickened as shown in (a) and (b), respectively, to make the orthogonal representation $H_A(\phi)$ biconnected.

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