Università degli Studi di Roma Tre
Dipartimento di Informatica e Automazione
Via della Vasca Navale, 79 – 00146 Roma, Italy

Non-Convex Representations of Graphs

Giuseppe Di Battista, Fabrizio Frati, and Maurizio Patrignani

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Dipartimento di Informatica e Automazione,
Università Roma Tre,
Rome, Italy.
{gdb,frati,patrigna}@dia.uniroma3.it

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ABSTRACT

We show that every plane graph admits a planar straight-line drawing in which all faces with more than three vertices are non-convex polygons.

1 Introduction

In a *straight-line planar drawing* of a graph each edge is drawn as a segment and no two segments intersect. A *convex drawing* is a straight-line planar drawing in which each face is a convex polygon. While every plane graph admits a planar straight-line drawing [16, 6, 13], not every plane graph admits a convex drawing. Tutte showed that every triconnected plane graph admits such a drawing with its outer face drawn as an arbitrary convex polygon [15]. Thomassen [14] characterized the graphs admitting a convex drawing and Chiba *et al.* [3] presented a linear-time algorithm for producing such drawings. Convex drawings can be efficiently constructed in small area [4, 10, 5, 1, 2].

As opposed to traditional convex graph drawing, several research works explored the properties of drawings with non-convexity requirements.

Hong and Nagamochi proved that every triconnected plane graph admits a planar straight-line drawing whose boundary is an arbitrary star-shaped polygon and every internal face is a convex polygon. Such a drawing can be obtained in linear time [8]. The same authors also gave necessary and sufficient conditions for a graph with a fixed planar embedding to admit a straight-line drawing where the internal faces are star-shaped and the angles greater than π are a subset of a pre-specified set of corners in the embedding [9].

A *pointed drawing* is a planar straight-line drawing such that each vertex is incident to a face where it forms an angle larger than π. In [7] it is shown that a planar graph admits a pointed drawing if and only if it is a planar Laman graph or a subgraph of a planar Laman graph. A *Laman graph* is an n-vertex graph with exactly $2n-3$ edges whose subgraphs with k vertices have at most $2k-3$ edges. The family of Laman graphs coincides [11] with that of minimally rigid graphs [12]. Roughly, a *rigid graph* is a planar graph that admits a straight-line drawing whose vertex positions cannot be slightly perturbed while preserving edge lengths (up to rotations and translations of the whole drawing). A *minimaly rigid graph* is a rigid graph where no edge can be removed without loosing rigidity.

A *pseudo-triangle* is a simple polygon with exactly three angles smaller than π. A *pseudo-triangulation* is a planar straigth-line drawing of a graph where all bounded faces are pseudo-triangles. In [7] it is proved that the class of graphs that can be embedded as pointed pseudo-triangulations coincides with the one of planar minimally-rigid graphs. Further, in the same paper it is conjectured that the graphs that can be drawn as pseudo-triangulations are exactly the planar generically-rigid graphs. The relationship among the families of graphs admitting pointed drawings, drawings as pointed pseudo-triangulations, and drawings as pseudo-triangulations is graphically depicted in Fig. 1.

In this paper we address the problem of producing *non-convex drawings*, i.e., drawings where all faces with more than three vertices are non-convex. This can be considered as the opposite of the classic problem of constructing convex drawings. Also, it can be seen as the dual of the problem of constructing pointed drawings, since faces, and not vertices, are constrained to have an angle greater than π. We prove the following:

Theorem 1 *Every plane graph admits a non-convex drawing.*

In Sect. 3 we prove the previous theorem for biconnected graphs by means of a constructive algorithm whose inductive approach is reminescent of Fary's construction [6], although it applies to non-triangulated graphs and relies on a more complex case study. In Sect. 4 we discuss how to extend the result to general plane graphs.

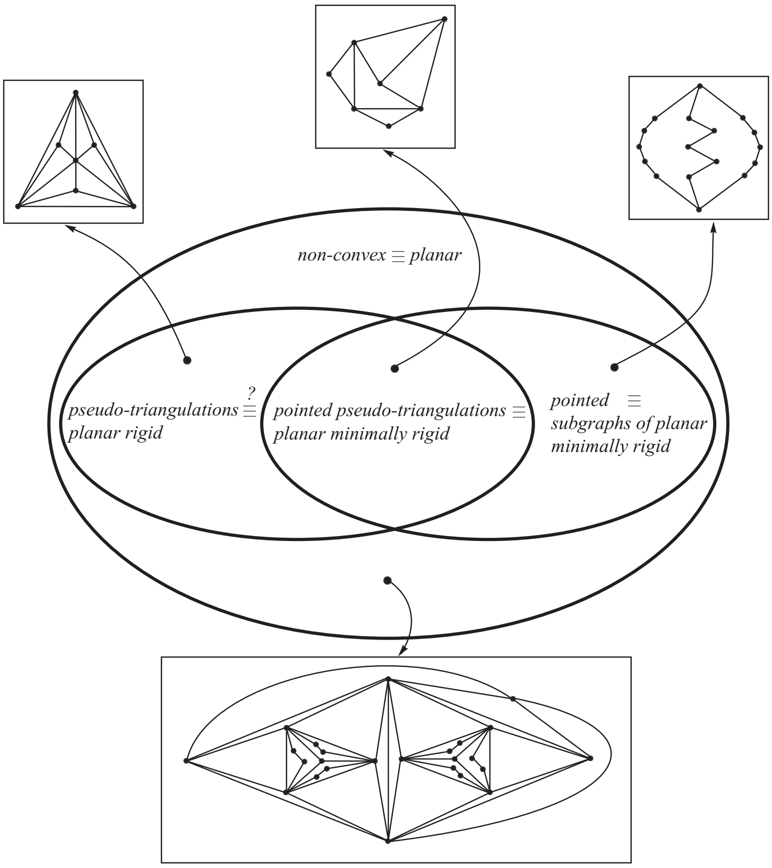


Figure 1: Inclusion among graph classes admitting non-convex drawings, pointed drawings, drawings as pointed pseudo-triangulations, and drawings as pseudo-triangulations.

2 Preliminaries

A graph is *simple* if it has no multiple edges and no self-loops. A *k-connected* graph G is such that removing any $k-1$ vertices leaves G connected; 3-connected, 2-connected, and 1-connected graphs are also called *triconnected*, *biconnected*, and *simply connected* graphs, respectively. A *separating k-set* of a graph G is a set of k vertices whose removal increases the number of connected components of G. Separating 1-sets and separating 2-sets are *cutvertices* and *separation pairs*, respectively. The connected components obtained from the removal of a cutvertex c are called the *blocks incident to c*. A separating k-cycle is a simple cycle of k vertices whose removal disconnects the graph.

A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of a graph are *equivalent* if they determine the same circular ordering around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into *faces*. The unbounded face is the *outer face*. In the following the outer face of a graph G is denoted by $f(G)$. A *chord* of $f(G)$ is an edge connecting two non-consecutive vertices of $f(G)$. A graph with a planar embedding and an outer face is called a *plane graph*.

Consider a face f of a plane graph G and two non-adjacent vertices u and v incident to f. The *contraction of u and v inside f* leads to a graph G' in which u and v are replaced by a single vertex w, connected to all vertices u and v are connected to in G. If

u and v are connected to the same vertex y, then G' contains two edges (w, y) (hence G' is not simple), unless (u, y) and (y, v) are incident to f (in this case there is only one edge (w, y)). The contraction operation is also defined on two adjacent vertices u and v, by removing edge (u, v) and contracting u and v inside the face created by the removal. In the following we only perform contraction operations that preserve the simplicity of the graph.

A *non-convex drawing* of a plane graph G is a planar straight-line drawing of G in which each face f with more than three vertices has an angle greater than π (smaller than π) if f is an internal face (the outer face, respectively).

3 Biconnected Plane Graphs

In this section we prove Theorem 1 for biconnected graphs. Hence, throughout the section, we assume that the input graph G is biconnected. An extension of the proof to simply connected graphs is shown in the next section.

First, we observe that the following lemmata hold.

Lemma 1 *Suppose that G has a face f with at least four incident vertices. Then there exist two vertices u and v incident to f, such that edge (u, v) can be added to G inside f, so that the resulting plane graph G' is simple.*

Proof: Since f is a face of G, the subgraph of G induced by the vertices incident to f is outerplanar, hence it does not contain K_4 as a subgraph. It follows that among any four vertices incident to f there exist two vertices u and v that are not joined by any edge of G. Edge (u, v) can hence be added to G inside f, proving the lemma. □

Lemma 2 *Let G be a plane graph and let f be a face with more than four incident vertices. Let G' be the plane graph obtained by inserting inside f an edge e between two non-adjacent vertices of f. Suppose that a non-convex drawing Γ' of G' exists. Then the drawing Γ obtained by removing e from Γ' is a non-convex drawing of G.*

Proof: Each face of G' with more than three incident vertices has an angle greater than π in Γ'. All faces of G, but for f, are represented in Γ by the same polygons by which they are represented in Γ'. Hence, we have only to prove that if f is an internal face (the outer face) then f contains an angle greater than π (smaller than π, respectively).

Suppose f is internal. Let f_1 and f_2 be the faces that are introduced in G' by splitting f with the insertion of edge e in G. Since f has more than four incident vertices, then at least one out of f_1 and f_2 has more than three incident vertices. Without loss of generality, let f_1 be such a face. Since Γ' is non-convex, at least one of the internal angles of f_1, say α, is more than π. If α is not incident to any end-vertex of e, then f contains exactly the same angle in Γ. Otherwise, α is incident to an end-vertex of e. Removing e from Γ' leads to an angle α' strictly greater than α.

Suppose f is the outer face (i.e. $f \equiv f(G)$). Then, one out of f_1 and f_2, say f_1, is the outer face of G', while f_2 is an internal face of G'. Since every simple polygon has at least three internal angles smaller than π and since exactly two angles internal to f_2 are incident to e, then f_2 contains at least one angle which is smaller than π and which is not incident to e. Such an angle is also an angle of f. □

In order to prove Theorem 1 for biconnected plane graphs, we prove the following:

Theorem 2 *Let G be a biconnected plane graph such that all the faces of G have three or four incident vertices. Let $f(G)$ be the outer face of G.*

- *If $f(G)$ has three vertices, then, for every triangle T in the plane, G admits a non-convex drawing in which $f(G)$ is represented by T.*
- *If $f(G)$ has four vertices and no chord, then, for every non-convex quadrilateral Q in the plane, G admits a non-convex drawing in which $f(G)$ is represented by Q.*

The mapping of the vertices of $f(G)$ to the vertices of T or Q is arbitrary, provided that the circular ordering of the vertices of $f(G)$ is respected.

Theorem 2, together with Lemmata 1 and 2, implies Theorem 1 for biconnected graphs. Namely, let G be any biconnected plane graph. While G has any face f with at least five vertices, add, by Lemma 1, a dummy edge inside f so that the augmented graph is still plane and simple. The obtained graph G' has (at least) one face with four incident vertices for each face of G with more than three incident vertices. In order to apply Theorem 2, consider $f(G')$ and, if $f(G')$ has four incident vertices and has a chord, insert a dummy edge inside $f(G')$ between the two vertices not incident to the chord, turning $f(G')$ into a triangular face. Construct a non-convex drawing of G', as in the proof of Theorem 2. By Lemma 2, removing the inserted dummy edges leaves the drawing non-convex.

Before proving Theorem 2, we give two more lemmata. Let G_5^* be the plane graph having a 4-cycle (u_1, u_2, u_3, u_4) as outer face and one internal vertex u_5 connected to u_1, u_2, u_3, and u_4 (see Fig. 2.a). Let G_6^* be the plane graph having a 4-cycle (u_1, u_2, u_3, u_4) as outer face and two connected internal vertices u_5 and u_6 with u_5 connected to u_1 and u_2, and u_6 connected to u_3 and u_4 (see Fig. 2.b).

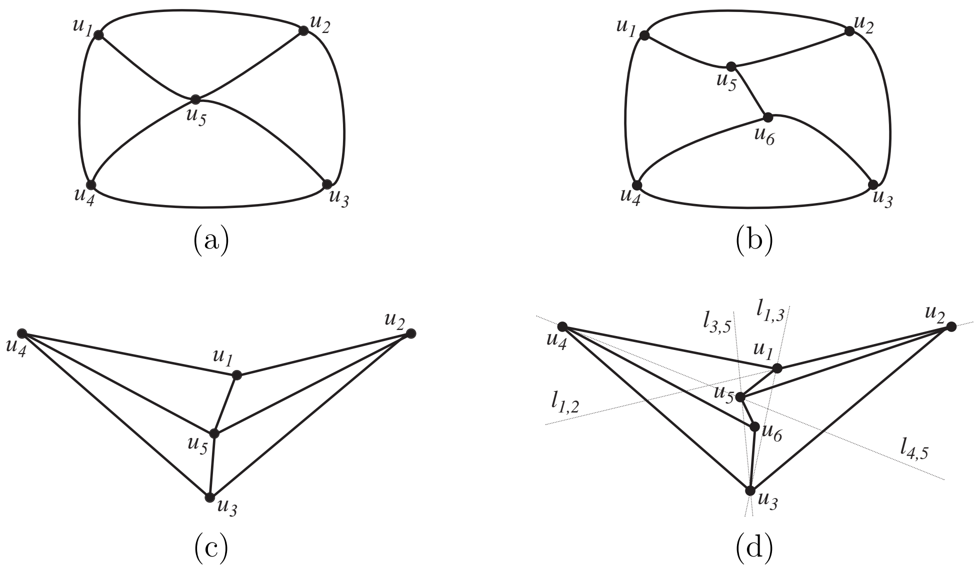


Figure 2: (a) G_5^*. (b) G_6^*. (c) Construction of a non-convex drawing of G_5^*. (d) Construction of a non-convex drawing of G_6^*.

Lemma 3 *For any non-convex quadrilateral Q in the plane, there exists a non-convex drawing of G_5^* such that $f(G_5^*)$ is represented by Q. The mapping of the vertices of $f(G_5^*)$ to the vertices of Q is arbitrary, provided that the circular ordering of the vertices of $f(G_5^*)$ is respected.*

Proof: Refer to Fig. 2.c. Let Q be any non-convex quadrilateral in the plane. Let p_1, p_2, p_3, and p_4 be the clockwise order of the vertices of Q, where u_i is drawn on p_i, for $i = 1, 2, 3, 4$. Choose p_5 to be any point inside Q such that straight-line edges can be drawn to the vertices of Q without crossing Q. The set of points with such a property is known as the *kernel* of Q. Any quadrilateral has a non-empty kernel, hence a point p_5 with the above properties can be found. Mapping vertex u_5 to p_5 yields a non-convex drawing of G_5^*. □

Lemma 4 *For any non-convex quadrilateral Q in the plane, there exists a non-convex drawing of G_6^* such that $f(G_6^*)$ is represented by Q. The mapping of the vertices of $f(G_6^*)$ to the vertices of Q is arbitrary, provided that the circular ordering of the vertices of $f(G_6^*)$ is respected.*

Proof: Refer to Fig. 2.d. Let Q be any non-convex quadrilateral in the plane. Let p_1, p_2, p_3, and p_4 be the clockwise order of the vertices of Q, where u_i is drawn on p_i, for $i = 1, 2, 3, 4$. Suppose that the non-convex angle of Q is incident to p_1, the other cases being analogous. Let $l_{i,j}$ denote the line through points p_i and p_j. Choose p_5 to be any point inside Q, in the half-plane delimited by $l_{1,3}$ and containing p_4, and in the half-plane delimited by $l_{1,2}$ and containing p_3. Map u_5 to p_5. Choose p_6 to be any point inside Q, in the half-plane delimited by $l_{4,5}$ and containing p_3, and in the half-plane delimited by $l_{3,5}$ and containing p_2. Map u_6 to p_6. See Fig. 2.b. The resulting drawing is a non-convex drawing of G_6^*. □

Proof of Theorem 2: Let G be a biconnected plane graph having all faces of size three or four and such that, if $f(G)$ has size four, then it has no chord.

The proof is by induction on the number of internal vertices of G. In the base case, either G has no internal vertex and the statement is trivially true, or $G = G_5^*$, or $G = G_6^*$. In the latter cases the statement follows from Lemmata 3 and 4, respectively. Inductively assume that the statement holds for any biconnected plane graph with less than n internal vertices. Suppose that G has n internal vertices. Three are the cases.

***G* has a separating *3*-cycle *C*.** Denote by G_1 the graph obtained by removing from G all vertices internal to C. Notice that G_1 has less than n internal vertices, that G_1 is simple and biconnected, and that C delimits an internal face of G_1. Further, if $f(G)$ has three incident vertices (if $f(G)$ has four incident vertices and has no chords), then $f(G_1)$ has three incident vertices (resp. $f(G_1)$ has four incident vertices and has no chords). Denote by G_2 the subgraph of G induced by the vertices internal to and on the border of C. Notice that G_2 has less than n internal vertices, that G_2 is simple and biconnected, and that C delimits the outer face of G_2, that hence has three incident vertices.

If $f(G_1)$ has three (four) incident vertices, then consider any triangle T (resp. any non-convex quadrilateral Q) and apply the inductive hypothesis to construct a non-convex drawing Γ_1 of G_1 having T (resp. Q) as outer face. Consider the triangle T' representing C in Γ_1. Apply the inductive hypothesis to construct a non-convex drawing Γ_2 of G_2 having T' as outer face and insert Γ_2 inside Γ_1 by gluing the two drawings along the common face C represented by T' in both drawings. The resulting drawing Γ is a non-convex drawing of G.

***G* has no separating *3*-cycle and *G* has a separating *4*-cycle *C*.** Denote by G_1 the graph obtained by removing from G all vertices internal to C. Notice that G_1 has less than n internal vertices, that G_1 is simple and biconnected, and that C delimits an internal face of G_1. Further, if $f(G)$ has three incident vertices (if $f(G)$ has four incident vertices and has no chords), then $f(G_1)$ has three incident vertices (resp. $f(G_1)$ has four incident vertices and has no chords). Denote by G_2 the subgraph of G induced by the vertices internal to and on the border of C. Notice that G_2 has less than n internal vertices, that G_2 is simple and biconnected, and that C delimits the outer face of G_2. Hence, $f(G_2)$ has four incident vertices. However, $f(G_2)$ has no chords, otherwise G_2 (and then G) would contain a separating 3-cycle.

If $f(G_1)$ has three (four) incident vertices, then consider any triangle T (resp. any non-convex quadrilateral Q) and apply the inductive hypothesis to construct a non-convex drawing Γ_1 of G_1 having T (resp. Q) as outer face. Consider the non-convex quadrilateral Q' representing C in Γ_1. Apply the inductive hypothesis to construct a non-convex drawing Γ_2 of G_2 having Q' as outer face and insert Γ_2 inside Γ_1 by gluing the two drawings along the common face C represented by Q' in both drawings. The resulting drawing Γ is a non-convex drawing of G.

***G* has no separating *3*-cycle and no separating *4*-cycle.** We are going to contract two vertices of G to obtain a graph with $n-1$ vertices. In order to apply the inductive hypothesis, the two vertices to contract must be chosen so that the graph resulting from the contraction is simple, is biconnected, and has no chords if the outer face has size four.

Consider any internal vertex v of G. First, we show that at least one of the following statements holds: (i) there exist two triangular faces incident to v and sharing an edge (v,u), such that contracting edge (v,u) does not create chords of $f(G)$; (ii) there exists a quadrilateral face $f=(v,u_1,u,u_2)$, such that either contracting v and u or contracting u_1 and u_2 does not create chords of $f(G)$; (iii) G is G_5^*; (iv) G is G_6^*.

- First, suppose that two triangular faces (u,v,u_1) and (u,v,u_2) exist. If also u is internal to G, then contracting (u,v) does not create chords of $f(G)$ and statement (i) holds.

  Otherwise, assume that u is incident to $f(G)$. Then, statement (i) does not hold only if both the following are true: (a) $f(G)$ is a 4-cycle and (b) there exists an edge connecting v and the vertex w of $f(G)$ not adjacent to u. In fact, if (a) does not hold, then $f(G)$ is a 3-cycle and no chord can be generated by contracting any two vertices. Also, if (b) does not hold, then contracting v to u does not create chords of $f(G)$.

  Hence, if statement (i) does not hold, $f(G)$ is a 4-cycle (u,w_1,w,w_2) and edge (v,w) exists. If one of u_1 and u_2 is internal to G, then either (u,v,w,w_1) or (u,v,w,w_2) is a separating 4-cycle, contradicting the hypotheses. Hence, both u_1 and u_2 are incident to $f(G)$. The four 3-cycles (u,v,u_1), (u,v,u_2), (w,v,u_1), and (w,v,u_2) do not have internal vertices by hypothesis. Hence, G contains no internal vertex other than v, so $G=G_5^*$, and statement (iii) holds.

- If two triangular faces incident to v and sharing an edge do no exist, then there exists a face $f=(v,u_1,u,u_2)$. Edge (u,v) does not belong to G, otherwise either (v,u,u_1) or (v,u,u_2) would be a separating 3-cycle. Analogously, edge (u_1,u_2) does not belong to G. If u is internal to G, then contracting v and u in f does not create chords of $f(G)$ and statement (ii) holds.

Otherwise, with similar arguments as above, statement (ii) does not hold for v and u only if both the following are true: (a) $f(G)$ is a 4-cycle and (b) there exists an edge connecting v and the vertex w of $f(G)$ not adjacent to u.

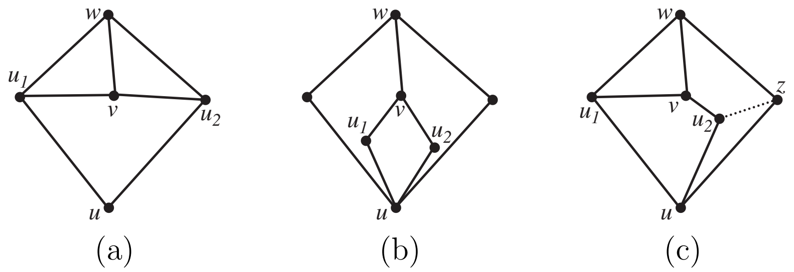


Figure 3: (a) Both u_1 and u_2 are incident to $f(G)$. (b) Both u_1 and u_2 are internal vertices of G. (c) Vertex u_1 is incident to $f(G)$ and vertex u_2 is not.

Hence, if statement (ii) does not hold, $f(G)$ is a 4-cycle and edge (v, w) exists. We distinguish three cases:

- If both u_1 and u_2 are incident to $f(G)$ (see Fig. 3.a), then G contains 3-cycles (u_1, v, w) and (u_2, v, w). Since G has no separating 3-cycle, such cycles are faces of G, contradicting the hypothesis that two triangular faces incident to v and sharing an edge do no exist.
- If both u_1 and u_2 are internal vertices of G (see Fig. 3.b), then contracting u_1 and u_2 does not create chords of $f(G)$ and statement (ii) holds.
- If one out of u_1 and u_2, say u_1, is incident to $f(G)$ and the other, say u_2, is not (see Fig. 3.c), consider the fourth vertex z of $f(G)$. If edge (u_2, z) does not exist, then vertices u_1 and u_2 can be contracted without creating chords of $f(G)$ and statement (ii) holds. If edge (u_2, z) exists, then there exist 3-cycles (u_1, v, w) and (u, u_2, z), and 4-cycles (v, u_1, u, u_2) and (v, w, z, u_2). Such cycles contain no vertices in their interior, since G has no separating 3-cycle or 4-cycle. Hence, $G = G_6^*$ and statement (iv) holds.

We prove that whichever of the statements holds, a non-convex drawing of G can be constructed.

Suppose that statement (i) holds, with G having faces (v, u, w_1) and (v, u, w_2). Contract edge (u, v) to a vertex w. We prove that the resulting graph G' satisfies the inductive hypotheses.

- If the outer face of G' has four vertices, then it has no chords by the definition of statement (i).
- G' is simple. If not, then there would exist a vertex w_3 adjacent to both u and v in G, with $w_3 \neq w_1, w_2$. This would imply that (u, v, w_3) is a separating 3-cycle, contradicting the hypotheses.
- All faces of G' have three or four incident vertices. Each face of G' that is not incident to w is also a face of G; further, each face of G' that is incident to w can be obtained from a face of G incident to exactly one of u and v by replacing such a vertex with w. Notice that no face $f_{u,v}$ incident to both u and v exists in of G other

than (v,u,w_1) and (v,u,w_2). In fact, provided that $f_{u,v}$ has at most four incident vertices, the existence of $f_{u,v} \neq (v,u,w_1),(v,u,w_2)$ would imply that there exists a vertex w_3 adjacent to both u and v in G, with $w_3 \neq w_1, w_2$, which we have already observed to be a contradiction.

- G' is biconnected. Suppose, for a contradiction, that w is a cutvertex of G'. It is easy to observe that the other vertices of G' can not be cutvertices, otherwise they would be cutvertices in G, as well. However, if w is a cutvertex, then (u,v) is a separating pair of G. Hence, the removal of u and v splits G in k connected components $G_1, G_2, \cdots, G_k$, with $k \geq 2$. Denote by $G_i^{u,v}$ the plane subgraph of G induced by the vertices of G_i, by u, and by v. Vertices w_1 or w_2 can not be in the same component $G_i^{u,v}$, otherwise (u,v) would not be a separating pair. In fact (see Fig. 4.a), no component $G_j^{u,v} \neq G_i^{u,v}$ can exist without violating the planarity of G. Hence, there exist distinct components $G_i^{u,v}$ and $G_j^{u,v}$ containing w_1 and w_2,

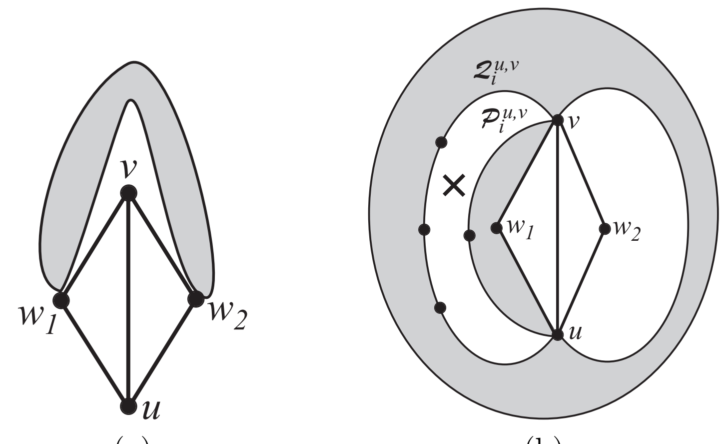


Figure 4: (a) If w_1 and w_2 belong to the same connected component, then (u,v) would not be a separating pair. (b) The face of G incident to $\mathcal{P}_i^{u,v}$ and $\mathcal{Q}_i^{u,v}$ and marked with a cross contains at least five vertices. Notice that u and/or w_2 could be incident to the outer face of G.

respectively. Either $G_i^{u,v}$ does not contain $G_j^{u,v}$ inside one of its internal faces or vice-versa. Suppose without loss of generality that the first is true. Then, the outer face of $G_i^{u,v}$ consists of edge (u,v) and of a path $\mathcal{P}_i^{u,v}$ connecting u and v. Such a path contains at least one vertex different from u and v, otherwise (u,v) would be a multiple edge. Since v is an internal vertex of G, there exists an internal face of G that is composed by two paths between u and v, one of which is $\mathcal{P}_i^{u,v}$ and the other one of which is a path $\mathcal{Q}_i^{u,v}$ (see Fig. 4.b). Path $\mathcal{Q}_i^{u,v}$ contains at least three vertices different from u and v, otherwise G would contain a separating 3-cycle or a separating 4-cycle (namely $\mathcal{Q}_i^{u,v} \cup (u,v)$ would contain w_1 in its interior). Then, the face of G incident to $\mathcal{P}_i^{u,v}$ and $\mathcal{Q}_i^{u,v}$ contains at least five vertices, contradicting the assumptions.

Inductively construct a non-convex drawing Γ' of G' (see Fig. 5.a). Now consider the point p where vertex w is drawn in Γ'. There exists a small disk D (see Fig. 5.b) centered at p such that moving p to any point inside D leaves Γ' a non-convex drawing. Consider the line l' through p and orthogonal to the line l connecting u_1 and u_2. Remove w and its incident edges. Insert vertices u and v on l', so that both are inside D. Connect u and v to their neighbors. It is easy to see that all faces that had an angle greater than π in Γ' still have an angle greater than π in Γ and that the drawing is still planar.

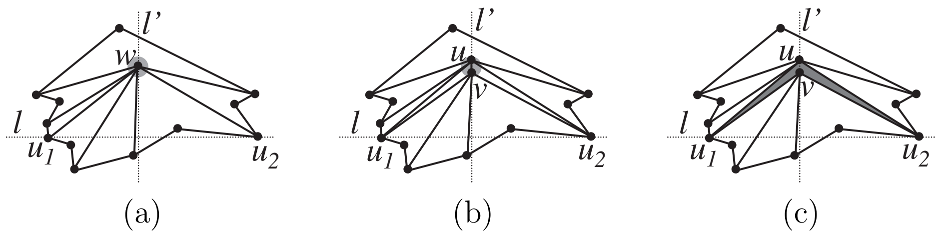


Figure 5: (a) and (b): Inductive construction of Γ, when statement (i) holds. The light-shaded region represents disk D. (a) and (c): Inductive construction of Γ, when statement (ii) holds. The dark-shaded region represents face (u, u_1, v, u_2) in Γ.

Now, suppose that statement (ii) holds. More precisely, suppose that contracting u and v inside a face (u, u_1, v, u_2) does not create chords of $f(G)$ (the case in which vertices u_1 and u_2 can be contracted is analogous). Contract u and v inside (u, u_1, v, u_2) to a vertex w. We prove that the resulting graph G' satisfies the inductive hypotheses.

- If the outer face of G' has four vertices, then it has no chords by the definition of statement (ii).

- G' is simple. If not, then there would exist a vertex u_3 adjacent to both u and v in G, with $u_3 \neq u_1, u_2$. This would imply that (u, u_1, v, u_3) or (u, u_2, v, u_3) is a separating 4-cycle, contradicting the hypotheses.

- All faces of G' have three or four incident vertices. Each face of G' that is not incident to w is also a face of G; further, each face of G' that is incident to w can be obtained from a face of G incident to exactly one of u and v by replacing such a vertex with w. Notice that no face $f_{u,v}$ incident to both u and v exists in G other than (u, u_1, v, u_2). In fact, provided that $f_{u,v}$ has at most four incident vertices, the existence of $f_{u,v} \neq (u, u_1, v, u_2)$ would imply that there exists a vertex u_3 adjacent to both u and v in G, with $u_3 \neq u_1, u_2$, which we have already observed to be a contradiction.

- G' is biconnected. As in the proof of the biconnectivity of the graph resulting from the contraction of two adjacent vertices, it is possible to argue that: (i) it is sufficient to study whether w is a cutvertex of G', and hence whether (u, v) is a separating pair of G; (ii) vertices u_1 or u_2 are in two distinct components $G_i^{u,v}$ and $G_j^{u,v}$, respectively, that are the plane subgraphs of G induced by u, by v, and by the vertices of two of the connected components $G_1, G_2, \cdots, G_k$ resulting from the removal of u and v, namely components G_i and G_j; (iii) the outer face of one out of $G_i^{u,v}$ and $G_j^{u,v}$, say $G_i^{u,v}$, consists of edge (u, v) and of a path $\mathcal{P}_i^{u,v}$ connecting u and v and containing at least one vertex different from u and v; and (iv) there exists an internal face of G that is composed by two paths between u and v, one of which is $\mathcal{P}_i^{u,v}$ and the other one of which is a path $\mathcal{Q}_i^{u,v}$. In order to complete the proof it is sufficient to observe that path $\mathcal{Q}_i^{u,v}$ contains at least two vertices different from u and v, otherwise G

would contain a separating 4-cycle (namely $\mathcal{Q}_i^{u,v} \cup (u, w_2) \cup (w_2, v)$ would contain w_1 in its interior). Then, the face of G incident to $\mathcal{P}_i^{u,v}$ and $\mathcal{Q}_i^{u,v}$ contains at least five vertices, contradicting the assumptions.

Inductively construct a non-convex drawing Γ' of G' (see Fig. 5.a). Perturb the vertices so that they are in general position and that Γ' is still a non-convex drawing. Consider the point p where w is drawn in Γ' and consider the line l' through p and orthogonal to the line l connecting u_1 and u_2 (see Fig. 5.c). There exists a small disk D centered at p such that D does not intersect l and such that moving p to any point inside D leaves Γ' non-convex. Remove w and its incident edges. Insert vertices u and v on l', so that both are inside D. Connect u and v to their neighbors. It is easy to see that all faces that had an angle greater than π in Γ' still have an angle greater than π in Γ and that the drawing is still planar. Further, face (u, v, u_1, u_2) is non-convex, as well, since the angle incident to the one of u and v closer to l is greater than π.

Finally, suppose that statement (iii) or (iv) holds. We are in one of the base cases and the claim directly follows from Lemmata 3 or 4, respectively. □

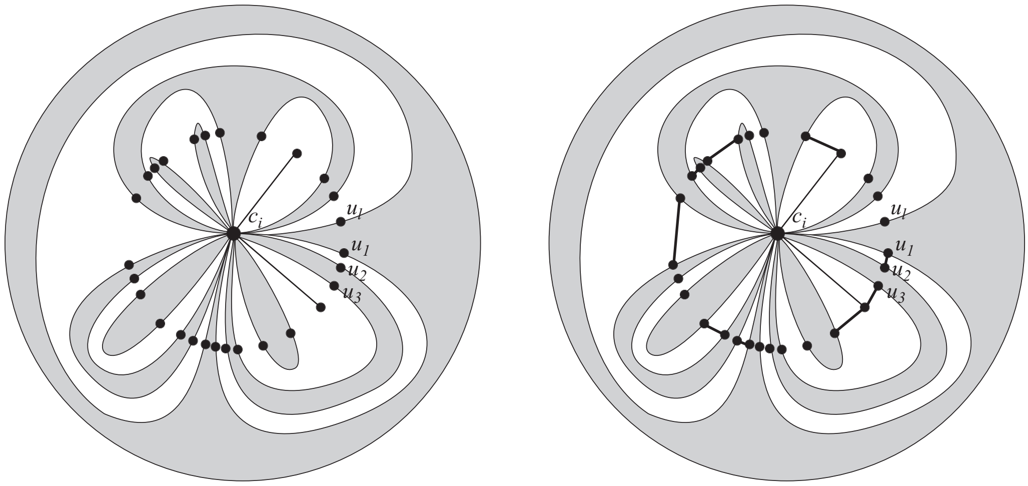
4 General Plane Graphs and Conclusions

We have proved that every biconnected plane graph admits a drawing in which each face with more than three vertices has an angle greater than π. In the following we extend the previous result to simply-connected plane graphs.

The extension works as follows: First, the simply-connected plane graph G given as input is augmented to a biconnected plane graph G' by adding dummy edges in such a way that, for each face of G with more than three vertices, there exists a corresponding face of G' with more than three vertices. Second, a non-convex drawing of G' is constructed by means of the algorithm shown in the previous section. Third, dummy edges inserted at the first step are removed from the drawing, turning the non-convex drawing of G' into a non-convex drawing of G.

Now we detail how to perform the previous claimed augmentation. Consider any simply-connected plane graph G. Suppose that G has k cutvertices $c_1, c_2, \dots, c_k$. We are going to augment G by adding dummy edges to it in k steps. During step i edges are added to G so that c_i is not longer a cutvertex. Consider the i-th step of the augmentation and consider vertex c_i. Refer to Fig. 6. Consider the clockwise order $u_1, u_2, \cdots, u_l$ of the neighbors of c_i starting at any vertex. Insert an edge between vertex u_j and vertex u_{j+1}, for each $j = 1, 2, \cdots, l-1$, if the following conditions are satisfied: (i) u_j and u_{j+1} belong to distinct blocks incident to c_i; (ii) no edge incident to a vertex of the block containing u_{j+1} has been added during step i of the augmentation. Edge (u_j, u_{j+1}) comes immediately before (u_j, c_i) in the clockwise order of the neighbors of u_j and immediately after (u_{j+1}, c_i) in the clockwise order of the neighbors of u_{j+1}. It is easy to see that the resulting graph is simple and planar, that c_i is not longer a cutvertex, and that after the augmentation there exists a face with at least three incident vertices for each face with at least three incident vertices before the augmentation, that is, during step i the augmentation does not triangulate any non-triangulated face of the graph before step i.

This concludes the proof that the case of simply-connected graphs can be reduced to that of biconnected graphs. Therefore, the family of graphs admitting a straight-line



(a) (b)

Figure 6: (a) The blocks around a cutvertex c_i before step i of the augmentation. The neighbors of c_i that are not incident to a face common to distinct blocks are not shown. (b) The edges added during step i of the augmentation.

planar drawing where all the faces with more than three vertices are non-convex polygons coincides with that of the planar graphs.

As open problems we suggest the following.

- Which is the time complexity of deciding if a plane graph admits a non-convex drawing where the angles greater than π are specified in advance?
- Which graphs admit a straight-line planar drawing which is simultaneously pointed and non-convex?
- Are there algorithms to construct non-convex drawings of planar graphs on a polynomial-size grid?

References

[1] I. Bárány and G. Rote. Strictly convex drawings of planar graphs. *Doc. Math.*, 11:369–391, 2006.

[2] N. Bonichon, S. Felsner, and M. Mosbah. Convex drawings of 3-connected plane graphs. *Algorithmica*, 47(4):399–420, 2007.

[3] N. Chiba, T. Yamanouchi, and T. Nishizeki. Linear algorithms for convex drawings of planar graphs. In J. A. Bondy and U. S. R. Murty, editors, *Progress in Graph Theory*, pages 153–173. Academic Press, New York, NY, 1984.

[4] M. Chrobak, M. T. Goodrich, and R. Tamassia. Convex drawings of graphs in two and three dimensions. In *Symposium on Computational Geometry*, pages 319–328, 1996.

[5] M. Chrobak and G. Kant. Convex grid drawings of 3-connected planar graphs. *Int. J. Comput. Geometry Appl.*, 7(3):211–223, 1997.

[6] I. Fary. On straight line representing of planar graphs. *Acta. Sci. Math.*, 11:229–233, 1948.

[7] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. L. Souvaine, I. Streinu, and W. Whiteley. Planar minimally rigid graphs and pseudo-triangulations. *Comput. Geometry Theory Appl.*, 31(1-2):31–61, 2005.

[8] S. Hong and H. Nagamochi. Convex drawings of graphs with non-convex boundary. In F.V. Fomin, editor, *WG 2006*, volume 4271 of *LNCS*, pages 113–124. Springer, 2006.

[9] S. Hong and H. Nagamochi. Star-shaped drawings of graphs with fixed embedding and concave corner constraints. In X. Hu and J. Wang, editors, *COCOON*, volume 5092 of *LNCS*, pages 405–414. Springer, 2008.

[10] G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16(1):4–32, 1996.

[11] G. Laman. On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, 4:331–340, 1970.

[12] G. Rote, F. Santos, and I. Streinu. *Pseudo-triangulations — a survey.* Contemporary Mathematics. American Mathematical Society, December 2008. to appear.

[13] S. K. Stein. Convex maps. *Proc. of the American Mathematical Society*, 2:464–466, 1951.

[14] C. Thomassen. Plane representations of graphs. In *Progress in Graph Theory*, pages 43–69. Academic Press, 1984.

[15] W. T. Tutte. Convex representations of graphs. *Proc. London Math. Soc.*, 10:304–320, 1960.

[16] K. Wagner. Bemerkungen zum vierfarbenproblem. *Jahresbericht. German. Math.-Verein*, 2:26–32, 1936.