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Upward Straight-line Embeddings of Directed Graphs into Point Sets

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ABSTRACT

In this paper we consider the problem of characterizing the directed graphs that admit an upward straight-line embedding into every point set in convex or in general position. In particular, we show that no biconnected directed graph admits an upward straight-line embedding into every point set in convex position, and we provide a characterization of the Hamiltonian directed graphs that admit upward straight-line embeddings into every point set in general or in convex position. We also describe how to construct upward straight-line embeddings of directed paths into convex point sets and we prove that for directed trees such embeddings do not always exist. Further, we investigate the related upward simultaneous embedding without mapping problem, proving that deciding whether two directed graphs admit an upward simultaneous embedding without mapping is $\mathcal{NP}$-hard.

1 Introduction

A *straight-line embedding* of a graph G into a point set P is a mapping of each vertex of G to a point of P and of each edge of G to a straight-line segment between its end-points such that no two edges intersect. The problem of constructing straight-line embeddings of graphs into planar point sets is well-studied from both a combinatorial and an algorithmic point of view and comes in several different flavours within the Graph Drawing literature.

Gritzmann *et al.* [11] proved that a graph admits a straight-line embedding into *every* point set in general position if and only if it is an *outerplanar graph*. From an algorithmic point of view, an $O(n \log^3 n)$-time algorithm [1] and a $\Theta(n \log n)$-time algorithm [2] are known for constructing straight-line embeddings of outerplanar graphs and trees into given point sets in general position, respectively. Cabello [4] proved that the problem of deciding whether a planar graph admits a straight-line embedding into a given point set is $\mathcal{NP}$-hard. If edges are not required to be straight then, by the results of Kaufmann and Wiese [12], every planar graph admits a planar drawing with at most two bends per edge into every point set and such a bound can not be improved.

Several problems ask for embedding more graphs on the same point set. Determining the minimum cardinality $f(n)$ of a set of points P such that every n-vertex planar graph admits a straight-line embedding into an n-point subset of P is a very well-know problem. The best known upper bound for $f(n)$ is quadratic [6], while only linear lower bounds are known [5, 13]. Recently, the problem of constructing simultaneous embeddings without mapping, i.e., straight-line planar drawings of n-vertex graphs on the same set of n points, has been considered. Brass *et al.* [3] proved that a planar graph and any number of outerplanar graphs admit a simultaneous embedding without mapping. Whether every two planar graphs admit a simultaneous embedding without mapping is still unknown.

Surprisingly, less attention has been devoted to the directed versions of such problems. When dealing with the visualization of directed graphs, one usually requires an *upward* drawing, i.e., a drawing such that each edge monotonically increases in the y-direction. Giordano, Liotta, Mchedlidze, and Symvonis [10] show a directed counterpart of the results in [12], namely that every upward planar digraph has an upward planar embedding with at most two bends per edge into every point set. As a directed counterpart of the results in [3], the same authors show that any number of trees admit an upward simultaneous embedding without mapping.

In this paper we obtain results on the directed versions of some of the problems cited above.

In Section 3, we deal with the problem of determining which directed graphs admit a planar straight-line upward embedding into every point set in general or in convex position. We show that no biconnected directed graph with more than three vertices has a straight-line upward embedding into every point set in convex position or into every point set in general position. We also characterize the Hamiltonian directed graphs that admit a straight-line upward embedding into every point set in convex or in general position. We prove that every directed path admits a straight-line upward embedding into every point set in convex position and that every directed tree of diameter at most four admits a straight-line upward embedding into every point set in convex or in general position. Finally, we prove that not all directed trees admit a straight-line upward embedding into every point set in convex position.

In Section 4, answering a question of Giordano *et al.* [10], we prove that there exist two upward planar directed graphs that do not admit any upward simultaneous embedding without mapping. Finally, we prove that deciding whether two upward planar digraphs admit an upward simultaneous embedding without mapping is $\mathcal{NP}$-hard.

2 Preliminaries

A *drawing* of a graph is a mapping of each vertex to a distinct point in the plane and of each edge to a Jordan curve between its endpoints. A *planar drawing* is such that no two edges intersect except, possibly, at common endpoints. A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same ordering around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. A *biconnected* (resp. *triconnected*) graph G is such that removing any vertex (resp. any two vertices) leaves G connected. A triconnected planar graph admits a unique planar embedding. An *outerplanar graph* admits a planar embedding in which all vertices are incident to the outer face. Such an embedding is called *outerplanar embedding*.

An *upward planar directed graph* is a directed graph that admits a planar drawing such that each edge is represented by a curve monotonically increasing in the y-direction. Every upward planar digraph admits a *straight-line* upward planar drawing [7], i.e., an upward planar drawing in which every edge is represented by a segment. The *underlying graph* of a directed graph G is the undirected graph obtained by removing the directions on the edges of G. In the following we refer to families of upward planar directed graphs as directed paths, directed cycles, directed trees, directed outerplanar graphs, meaning upward planar directed graphs whose underlying graphs are paths, cycles, trees, and outerplanar graphs, respectively. An *Hamiltonian* directed graph G is a directed graph containing a path $(v_1, v_2, \cdots, v_n)$ passing through all vertices of G such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for each $1 \leq i < n$. A *source* (resp. *sink*) in a directed graph G is a vertex having only outgoing edges, i.e., edges directed *from* the vertex to its neighbors (resp. having only incoming edges, i.e., edges directed from the neighbors *to* the vertex).

A set of points in the plane is in *general position* if no three points lie on the same line. The *convex hull* $H(P)$ of a set of points P is the set of points that can be obtained as a convex combination of the points of P. A set of points is in *convex position* if no point is in the convex hull of the others.

An *upward straight-line embedding* of an n-vertex directed graph G into a point set P with n points, is a mapping of each vertex of G to a point of P such that the resulting straight-line drawing is upward and planar[1]. An *upward simultaneous embedding without mapping* of two n-vertex upward planar directed graphs G_1 and G_2 is a pair of upward planar straight-line drawings in which the vertices of G_1 and G_2 are placed on the same set of n points.

In order to deal with upward embeddings of directed graphs into point sets, we assume that no two points of any point set have the same y-coordinate[2]. Then, the points of any point set P in general or convex position can be totally ordered by increasing y-coordinate. Hence, in a set of n points, we refer to the i-th point as to the point such that exactly $i-1$ points have smaller y-coordinate. The first and the last point of a point set P are denoted by $p_m(P)$ and $p_M(P)$, respectively. In a convex point set P two points are *adjacent* if the segment between

[1]Notice that the use of the word *embedding* is a bit misleading with respect to the definition of planar embedding. *Drawing* would be more appropriate than embedding, however, for consistency with existing literature, we talk about upward straight-line *embeddings* of graphs into point sets.

[2]Such an assumption does not lead to a great loss of generality as every point set can be turned into one without two points having the same y-coordinate, by rotating the Cartesian axes by an arbitrary-small constant ϵ. Further, assuming that no two points have the same y-coordinate avoids trivial counter-examples and the *a priori* impossibility of drawing an edge between two specified points of the point set.

them is on the border of the convex hull of P. Points $\{v_1, v_2, \cdots, v_k\}$ in a convex point set P are *consecutive* if v_i and v_{i+1} are adjacent, for each $1 \leq i < k$. We call a *one-side* convex point set any convex point set P in which $p_M(P)$ and $p_m(P)$ are adjacent.

3 Graphs with Upward Straight-line Embeddings into Every Point Set

In this section we consider the problem of determining which classes of directed graphs admit an upward straight-line embedding into every point set in general or in convex position.

3.1 Biconnected Directed Graphs and Hamiltonian Directed Graphs

First, we show that no biconnected directed graph with more than three vertices has an upward straight-line embedding into every point set in convex position, and hence into every point set in general position. The following two lemmata are well-known:

Lemma 1 *Let C be any directed cycle. The number of sources in C is equal to the number of sinks.*

Lemma 2 *Let O be a straight-line embedding of a directed graph into a point set in convex position. Then O is an outerplanar embedding.*

We now observe the following lemmata.

Lemma 3 *Let G be a directed graph containing a cycle C. Suppose that C has at least two vertices u and v that are sources in C. Then there exists a convex point set P such that G has no upward straight-line embedding into P.*

Proof: Consider any one-side convex point set P. Let s_1 and s_2 be two sources in C. Suppose, w.l.o.g., that the point of P on which s_1 is drawn has y-coordinate smaller than the one on which s_2 is drawn. Since s_2 is a source in C, there exists two edges (s_2, v_1) and (s_2, v_2) going out of s_2 and belonging to C. Suppose, w.l.o.g., that the point of P on which v_1 is drawn has y-coordinate smaller than the one of the point of P on which v_2 is drawn. Since C is a cycle, there exist two disjoint paths connecting v_1 and s_1. One of these paths does not contain s_2 and v_2 and hence it crosses edge (s_2, v_2). See Fig. 1.a. □

Lemma 4 *Let G be a directed graph containing a cycle C. Suppose that C has exactly one source s and one sink t. Suppose also that each of the two directed paths $\mathcal{P}_1$ and $\mathcal{P}_2$ of C connecting s and t has at least one node different from s and t. Then there exists a convex point set P such that G has no upward straight-line embedding into P.*

Proof: Consider any one-side convex point set P. Consider any drawing Γ of G into P and let $p(s)$ and $p(t)$ be the points of P where s and t are drawn in Γ, respectively. Consider the points $P_k = \{p_1 = p(s), p_2, \cdots, p_k = p(t)\}$ of P having y-coordinate greater or equal than the one of $p(s)$ and less or equal than the one of $p(t)$. Since the drawing is straight-line and upward, both $\mathcal{P}_1$ and $\mathcal{P}_2$ are completely inside the convex hull H_k of P_k. As both $\mathcal{P}_1$ and $\mathcal{P}_2$ touch the border of H_k in at least one point different from $p(s)$ and $p(t)$, $\mathcal{P}_1$ and $\mathcal{P}_2$ cross at least once. See Fig. 1.b. □

We obtain the following:

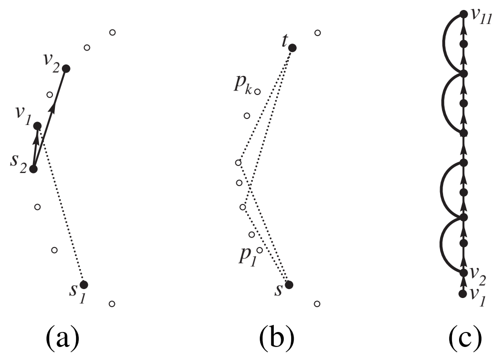


Figure 1: (a)–(b) Illustrations for the proofs of Lemmata 3 and 4, respectively. Dotted segments represent paths connecting two vertices. (c) A graph belonging to the family $\mathcal{G}_{11}$ of Hamiltonian directed graphs with 11 vertices that admit upward straight-line embeddings into every point set.

Theorem 1 *There exists no biconnected directed graph with more than three vertices that admits an upward straight-line embedding into every point set in convex position.*

Proof: Consider any biconnected directed graph G with more than three vertices. Consider any one-side convex point set P_1. By Lemma 2 any embedding Γ of G into P_1 is an outerplanar embedding, hence all vertices of G are incident to the outer face of G. Further, since G is biconnected, the outer face of G in Γ is a simple cycle C. Hence, C is a cycle passing through all vertices of G. By Lemma 3, C contains exactly one vertex s that is a source in C and hence, by Lemma 1, C contains exactly one vertex t that is sink in C. Further, by Lemma 4, at most one of the two directed paths $\mathcal{P}_1$ and $\mathcal{P}_2$ of C connecting s and t has at least one node different from s and t. It follows that one out of $\mathcal{P}_1$ and $\mathcal{P}_2$, say $\mathcal{P}_1$, is a Hamiltonian directed path and the other, say $\mathcal{P}_2$, is an edge (s, t). Now consider any convex point set P_2 that is not a one-side convex point set, i.e., any point set in which the line l connecting the first and the last point of P_2 determines two half-planes both containing points of P_2. Notice that such a point set exists if and only if $n \geq 4$. Since $\mathcal{P}_1$ is a Hamiltonian directed path between s and t, then there is a vertex of $\mathcal{P}_1$ in each point of P_2. It follows that there is at least one edge of $\mathcal{P}_1$ crossing l and hence crossing (s, t). □

Next, we characterize those Hamiltonian directed graphs that admit an upward straight-line embedding into every point set in general position and into every point set in convex position. Let $\mathcal{P}_n = (v_1, v_2, \cdots, v_n)$ be an n-vertex directed path, where edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $1 \leq i \leq n-1$. Let $\mathcal{G}_n$ be the family of n-vertex Hamiltonian directed graphs defined as follows: Each graph $G \in \mathcal{G}_n$ can be obtained by adding to $\mathcal{P}_n$ a set of edges E, where each edge of E is directed from a vertex v_i to a vertex v_{i+2}, for some $1 \leq i \leq n-2$, and no two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}) belong to E, for any $1 \leq i \leq n-3$.

Theorem 2 *An n-vertex Hamiltonian directed graph admits an upward straight-line embedding into every point set in general position if and only if it belongs to $\mathcal{G}_n$.*

Proof: A graph belonging to the family $\mathcal{G}_n$ of Hamiltonian directed graphs is shown in Fig. 1.c.

We first prove the necessity. Suppose that there exists an Hamiltonian directed graph G that admits an upward straight-line embedding into every point set and that does not belong to $\mathcal{G}_n$. If $\mathcal{P}_n = (v_1, v_2, \cdots, v_n)$ is the Hamiltonian directed path of G, then either G contains an edge

(v_i, v_j), with $i + 2 < j \leq n$, for some $1 \leq i \leq n-3$, or it contains two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}), for some $1 \leq i \leq n-3$.

Suppose G contains an edge (v_i, v_j), with $i + 2 < j \leq n$. Consider any convex point set P with the following property. Let l be the line through the i-th point and the j-th point of P. Then suppose P is such that the $(i+1)$-th point and the $(i+2)$-th point of P are on different sides of l. In order to satisfy the upward constraint, the k-th vertex of $\mathcal{P}_n$ has to be drawn on the k-th point of P, for $k = 1, \cdots, n$. It follows that path $(v_i, v_{i+1}, \cdots, v_j)$ crosses edge (v_i, v_j).

Suppose G contains two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}), for some $1 \leq i \leq n-3$. Consider any one-side convex point set P. In order to satisfy the upward constraint, the k-th vertex of $\mathcal{P}_n$ has to be drawn on the k-th point of P, for $k = 1, \cdots, n$. Then, edge (v_i, v_{i+2}) crosses edge (v_{i+1}, v_{i+3}).

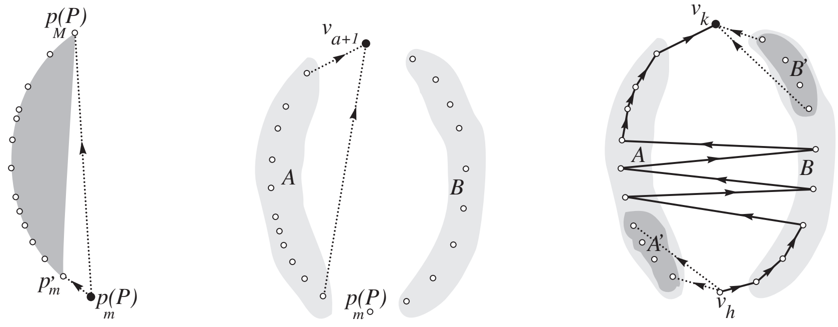
Next, we prove the sufficiency. Consider any n-vertex Hamiltonian directed graph G belonging to $\mathcal{G}_n$ and consider any point set P in general position. First, draw the Hamiltonian directed path $\mathcal{P}_n = (v_1, v_2, \cdots, v_n)$ of G into P as a y-monotone path. Hence, the k-th vertex of $\mathcal{P}_n$ is drawn at the k-th point of P, for $k = 1, \cdots, n$. The resulting drawing is planar since no two points have the same y-coordinate. Next, draw each edge (v_i, v_{i+2}) belonging to G. Since the drawing is straight-line, then each edge (v_i, v_{i+2}) can intersect or overlap only those edges that intersect the open horizontal strip S delimited by the horizontal lines through v_i and v_{i+2}. Since no two edges (v_i, v_{i+2}) and (v_{i+1}, v_{i+3}) belong to G, for any $1 \leq i \leq n-3$, then the only edges that have intersections with S are (v_i, v_{i+1}), (v_i, v_{i+2}) and (v_i, v_{i+2}). Since P is in general position, then v_i, v_{i+1}, and v_{i+2} are not on the same line and hence edges (v_i, v_{i+1}), (v_i, v_{i+2}) and (v_i, v_{i+2}) do not intersect or overlap. $\square$

3.2 Directed Paths and Directed Trees

We show how to construct upward straight-line embeddings of directed paths into point sets in convex position. We observe the following:

Lemma 5 *Let P be any one-side convex point set with n points and let $\mathcal{P} = (v_1, v_2, \cdots, v_n)$ be any n-vertex directed path. If edge (v_1, v_2) is directed from v_1 to v_2 (resp. from v_2 to v_1) then there exists an upward straight-line embedding of $\mathcal{P}$ into P in which v_1 is on $p_m(P)$ (resp. v_1 is on $p_M(P)$).*

Proof: We prove the statement by induction on the number of points of P (and of vertices of $\mathcal{P}$). If $n = 1$ the statement trivially follows. Consider any one-side convex point set P with n points, and any n-vertex directed path $\mathcal{P} = (v_1, v_2, \cdots, v_n)$. Suppose that (v_1, v_2) is directed from v_1 to v_2, the case in which (v_1, v_2) is directed from v_2 to v_1 being analogous. Consider the point set $P' = P \setminus \{p_m(P)\}$. Clearly, P' is a one-side convex point set. Namely, points $p_M(P') = p_M(P)$ and $p_m(P')$ are adjacent in P'. By the inductive hypothesis $\mathcal{P}' = (v_2, \cdots, v_n)$ admits an upward straight-line embedding into P' in which v_2 is either on $p_M(P')$ or on $p_m(P')$ (depending on the direction of edge (v_2, v_3)). In both cases v_1 can be mapped to $p_m(P')$ and edge (v_1, v_2) can be drawn as a segment. The resulting drawing Γ is straight-line by construction; further, Γ is upward, namely the upwardness of the drawing of $\mathcal{P}'$ comes by induction and edge (v_1, v_2) is directed upward, since $p_m(P)$ is the point of P with smallest y-coordinate; finally Γ is planar, namely the planarity of the drawing of $\mathcal{P}'$ comes by induction and edge (v_1, v_2) does not intersect any edge of $\mathcal{P}'$, since all the edges of $\mathcal{P}'$ are internal or on the border of the convex hull of the points in P', while edge (v_1, v_2) is external to such a convex hull. See Fig. 2.a. $\square$



(a) (b) (c)

Figure 2: (a) Illustration for the proof of Lemma 5. Dotted segments represent the two possible drawings of edge (v_1, v_2), in the case it is directed from v_1 to v_2. (b) Illustration for the proof of Theorem 3, when (v_a, v_{a+1}) is directed from v_a to v_{a+1}, and (v_{a+1}, v_{a+2}) is directed from v_{a+2} to v_{a+1}. Dotted segments represent the two possible drawings of edge (v_a, v_{a+1}). (c) Illustration for the proof of Theorem 3, when (v_a, v_{a+1}) is directed from v_a to v_{a+1}, and (v_{a+1}, v_{a+2}) is directed from v_{a+1} to v_{a+2}. Dotted segments represent the possible drawings of edges (v_{h-1}, v_h) and (v_k, v_{k+1}).

Theorem 3 *Every n-vertex directed path admits an upward straight-line embedding into every convex point set with n points.*

Proof: Let $\mathcal{P} = (v_1, v_2, \cdots, v_n)$ be any directed path and let P be any convex point set with n points. Let A and B be the subsets of P to the left and to the right, respectively, of the line through $p_M(P)$ and $p_m(P)$. Let $|A| = a$ and $|B| = b$. Consider edges (v_a, v_{a+1}) and (v_{a+1}, v_{a+2}).

If edge (v_a, v_{a+1}) is directed from v_a to v_{a+1} and (v_{a+1}, v_{a+2}) is directed from v_{a+2} to v_{a+1} (see Fig. 2.b), apply Lemma 5 to construct an upward straight-line embedding of path $\mathcal{P}_1 = (v_a, v_{a-1}, \cdots, v_1)$ into A in which v_a is placed either on $p_M(A)$ or on $p_m(A)$, and apply Lemma 5 to construct an upward straight-line embedding of path $\mathcal{P}_2 = (v_{a+1}, v_{a+2}, \cdots, v_n)$ into $B \cup \{p_M(P), p_m(P)\}$ in which v_{a+1} is placed on $p_M(P)$. The resulting drawing Γ is straight-line by construction; further, Γ is upward, namely the drawing of $\mathcal{P}_1$ and the drawing of $\mathcal{P}_2$ are upward by Lemma 5, and edge (v_a, v_{a+1}) is directed upward, since v_{a+1} is placed on $p_M(P)$; finally, Γ is planar, namely the drawings of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are planar by Lemma 5, they do not intersect each other because they lie in disjoint convex regions H_A and H_B (the convex hulls of point sets A and $B \cup \{p_M(P), p_m(P)\}$, respectively), and edge (v_a, v_{a+1}) does not intersect $\mathcal{P}_1$ or $\mathcal{P}_2$, since it lies outside H_A and outside H_B. An upward straight-line embedding of $\mathcal{P}$ into P can be constructed analogously if (v_a, v_{a+1}) is directed from v_{a+1} to v_a, and (v_{a+1}, v_{a+2}) is directed from v_{a+1} to v_{a+2}.

Now consider the case in which (v_a, v_{a+1}) is directed from v_a to v_{a+1}, and (v_{a+1}, v_{a+2}) is directed from v_{a+1} to v_{a+2} (see Fig. 2.c). Let h be the smallest index such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $i = h, h+1, \ldots, a$. Let k be the greatest index such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $i = a, a+1, \ldots, k-1$. Consider path $\mathcal{P}_1 = (v_{h-1}, v_{h-2}, \cdots, v_1)$ and consider the point set $A' \subseteq A$ composed of the first $h-1$ points of A. Apply Lemma 5 to construct an upward straight-line embedding of $\mathcal{P}_1$ into A' such that v_{h-1} is placed either on $p_M(A')$, or on $p_m(A')$. Consider path $\mathcal{P}_2 = (v_{k+1}, v_{k+2}, \cdots, v_n)$ and consider the point set $B' \subseteq B$ composed of the last $n-k$ points of B. Apply Lemma 5 to construct an upward straight-line embedding of $\mathcal{P}_2$ into B' such that v_{k+1} is placed either on $p_M(B')$, or on $p_m(B')$. Consider path $\mathcal{P}_3 = (v_h, v_{h+1}, \cdots, v_k)$ and consider the point set $C' \equiv P \setminus \{A' \cup B'\}$. Construct an upward straight-line embedding of $\mathcal{P}_3$ into C' such that the i-th vertex of $\mathcal{P}_3$ is placed on the i-th point of C', for $i = h, \cdots, k$. The resulting drawing Γ is straight-line by construction; further, Γ is upward, namely the drawing of $\mathcal{P}_1$ and the drawing of $\mathcal{P}_2$ are upward by Lemma 5, the drawing of $\mathcal{P}_3$ is upward by construction, edge (v_{h-1}, v_h), that by the minimality of h is directed from v_h to v_{h-1}, is directed upward, since v_h is placed on $p_m(P)$, and edge (v_k, v_{k+1}), that by the maximality of k is directed from v_{k+1} to v_k, is directed upward, since v_k is placed on $p_M(P)$; finally, Γ is planar, namely the drawings of $\mathcal{P}_1$ and of $\mathcal{P}_2$ are planar by Lemma 5, the drawing of $\mathcal{P}_3$ is planar because its vertices have increasing y-coordinates, the drawings of $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ do not intersect each other because they lie in disjoint convex regions $H_{A'}$, $H_{B'}$, and $H_{C'}$ (the convex hulls of point sets A', B', and C', respectively), and edges (v_{h-1}, v_h) and (v_k, v_{k+1}) do not intersect $\mathcal{P}_1$, $\mathcal{P}_2$, or $\mathcal{P}_3$, since they lie outside H_A, outside H_B, and outside H_C. An upward straight-line embedding of $\mathcal{P}$ into P can be constructed analogously if (v_a, v_{a+1}) is directed from v_{a+1} to v_a, and (v_{a+1}, v_{a+2}) is directed from v_{a+2} to v_{a+1}. $\square$

Directed trees of *diameter* at most four, i.e., in which the maximum number of edges in any path is at most four, admit upward straight-line embeddings into every point set in convex or in general position, as shown in the proof of the following theorem.

Theorem 4 *Every directed tree with diameter at most four admits an upward straight-line embedding into every point set in general position.*

Proof: Let T be any directed tree with diameter at most four. Consider any simple path $\mathcal{P}$ in T with the maximum number of edges (at most four, by definition of diameter). Root T at any node r of $\mathcal{P}$ with maximum distance from the extremes of $\mathcal{P}$ (e.g., if the diameter of $\mathcal{P}$ is four, the only node of $\mathcal{P}$ at distance two from the leaves of $\mathcal{P}$). Observe that each subtree of r has diameter one, i.e., is a *star*, hence it has a trivial upward straight-line embedding into every point set in general position (namely, place the root at the $(a+1)$-th point of the point set, where a is the number of edges ingoing the root, and draw straight-line edges from the root towards all other points). Let $b = \sum_i |T_i^{\downarrow}|$ be the total number of nodes in the subtrees $T_i^{\downarrow}$ of r whose root r_i has an edge directed from r_i to r. Place r at the $(b+1)$-th point $p(r)$ of P. Suppose that subtrees $T_i^{\downarrow}$ are ordered arbitrarily $T_1^{\downarrow}, T_2^{\downarrow}, \cdots, T_k^{\downarrow}$. Span the first b points of P by an half-line $l(r)$ fixed at point $p(r)$ rotating clockwise from an horizontal position. Subtree $T_1^{\downarrow}$ is drawn on the first $|T_1^{\downarrow}|$ points spanned by $l(r)$, subtree $T_2^{\downarrow}$ is drawn on the next $|T_2^{\downarrow}|$ points spanned by $l(r)$, and so on. An upward straight-line embedding of the subtrees $T_1^{\uparrow}, T_2^{\uparrow}, \cdots, T_l^{\uparrow}$ of r whose root r_i has an edge directed from r to r_i can be constructed analogously on the last $(n-b-1)$ points of P. The resulting drawing is straight-line by construction; further, it is upward since each subtree has an upward drawing and all the roots of the subtrees $T_1^{\downarrow}, T_2^{\downarrow}, \cdots, T_k^{\downarrow}$ (resp. $T_1^{\uparrow}, T_2^{\uparrow}, \cdots, T_l^{\uparrow}$) have y-coordinate smaller than the one of r (resp. greater than the one of r); finally it is planar, since each subtree has a planar drawing and each edge incident to the root does not intersect or overlap any other edges. Namely, two edges incident to the root can not intersect since they are straight-line or overlap since the point set is in general position; analogously, an edge incident to the root and to a vertex r_i, root of a subtree T_i, does not intersect or overlap any edge incident to r_i; further, an edge incident to the root and to a vertex r_i, root of a subtree T_i, does not intersect or overlap any edge incident to a vertex r_j, root of a subtree T_j, with $i \neq j$, since such edges lie in disjoint wedges of the plane. $\square$

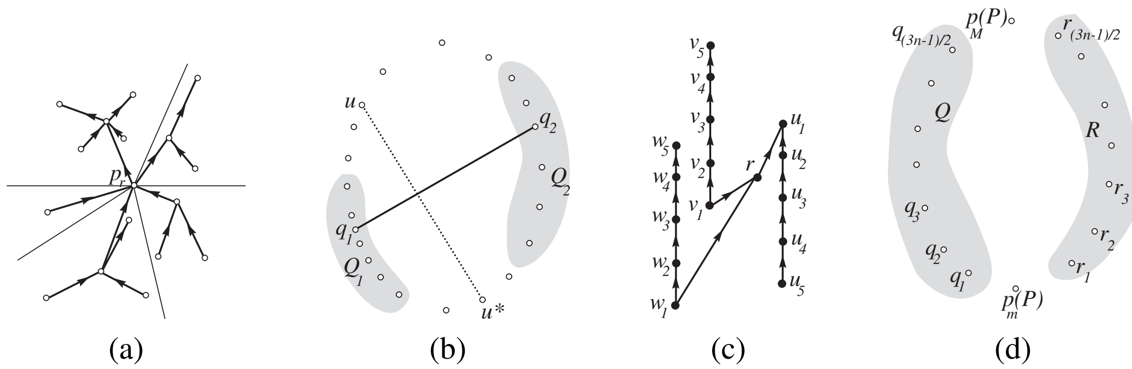


Figure 3: (a) Illustration for Theorem 4. (b) Illustration for the proof of Lemma 6. The dotted segment represents a path connecting vertices u and u^*. (c)–(d) A tree T and a point set P for the proof of Theorem 5.

Next, we show that not all directed trees admit a straight-line upward embedding into every point set in convex position (and hence into every point set in general position). The proof uses as main tool the following lemma. Consider any tree T and any convex point set P. Let u be any node of T and let $T_1, T_2, \cdots, T_k$ be the subtrees of T obtained by removing u and its incident edges from T.

Lemma 6 *In any upward straight-line embedding of T into P, the vertices of T_i are mapped into a set of consecutive points of P, for each $i = 1, 2, \cdots, k$.*

Proof: The lemma holds trivially if $k = 1$. Suppose $k \geq 2$ and suppose, for a contradiction, that there exists an upward straight-line embedding of T into P such that the vertices of a subtree T_i of T are mapped into a set of not-all-consecutive points of P. This implies that there exist two subsets Q_1 and Q_2 of P such that: (i) the points of Q_1 are consecutive in P, (ii) the points of Q_2 are consecutive in P, (iii) the points of $Q_1 \cup Q_2$ are not consecutive in P, (iv) there exists an edge between a point q_1 of Q_1 and a point q_2 of Q_2. The line through q_1 and q_2 splits the point set into two subsets P_1 and P_2. Assume, w.l.o.g., that vertex u is in P_1. Not only vertices of T_i are mapped into the points of P_2, otherwise the points of $Q_1 \cup Q_2$ would be consecutive in P. Hence, there is a vertex u^* of a tree $T_j \neq T_i$, that is placed on a vertex of P_2. Such a vertex is connected to u by a path composed of straight-line segments with endpoints at the points of P. Such a path intersects segment $\overline{q_1 q_2}$, providing a contradiction. See Fig. 3.b. $\square$

Assume n is odd and greater or equal than 5. Consider a directed tree T composed of: (i) one vertex r of degree three, (ii) three paths of n vertices $\mathcal{P}_1 = (u_1, u_2, \cdots, u_n)$, where (u_i, u_{i+1}) is directed from u_{i+1} to u_i, for $i = 1, 2, \cdots, n-1$, $\mathcal{P}_2 = (v_1, v_2, \cdots, v_n)$, where (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for $i = 1, 2, \cdots, n-1$, and $\mathcal{P}_3 = (w_1, w_2, \cdots, w_n)$, where (w_i, w_{i+1}) is directed from w_i to w_{i+1}, for $i = 1, 2, \cdots, n-1$, and (iii) edge (r, u_1) directed from r to u_1, edge (r, v_1) directed from v_1 to r, and edge (r, w_1) directed from w_1 to r. See Fig. 3.c. Let P be a convex point set with $3n+1$ points such that a set Q of $(3n-1)/2$ points $q_1, q_2, \cdots, q_{(3n-1)/2}$ are to the left and a set R of $(3n-1)/2$ points $r_1, r_2, \cdots, r_{(3n-1)/2}$ are to the right of the line connecting $p_M(P)$ and $p_m(P)$. Assume that $y(p_m(P)) < y(q_1) < y(r_1) < y(q_2) < y(r_2) < \cdots < y(q_{(3n-1)/2}) < y(r_{(3n-1)/2}) < y(p_M(P))$. See Fig. 3.d. It turns out that T does not admit any upward straight-line embedding into P.

Theorem 5 *For every n odd greater or equal than 5, there exists a $(3n+1)$-vertex directed tree T and a $(3n+1)$-point convex point set P such that T does not admit a straight-line upward embedding into P.*

Proof: Consider tree T and point set P described before the statement of the theorem. We show that every possible placement of r leads to a straight-line drawing of T that either is not upward or not planar.

First, we can exclude that r is placed on $p_M(P)$ or on $p_m(P)$, because there would be no point to place vertex u_1 or vertex v_1, respectively, without violating the upwardness of the drawing.

Suppose that r is placed on one of the first $(n+1)/2$ points of Q, say on a point $p(r)$. Then, by Lemma 6, one out of $\mathcal{P}_2$ and $\mathcal{P}_3$, say $\mathcal{P}_2$, is drawn on the points of Q below $p(r)$ (that are less or equal than $(n-1)/2$), on $p_m(P)$, and at least on the first $(n-1)/2$ points of R. It follows that there is no point of P with y-coordinate less than $p(r)$ and to which no vertex of $\mathcal{P}_2$ has been mapped to. Hence, there is no point to place w_1 without violating the upwardness of the drawing. Analogously, r can not be placed on one of the first $(n-1)/2$ points of R.

Now suppose that r is placed in any point of Q between the $\frac{n+1}{2}$-th and the $(n-1)$-th. Then, by Lemma 6, one out of $\mathcal{P}_2$ and $\mathcal{P}_3$, say $\mathcal{P}_2$, is drawn on the points of Q below $p(r)$ (that are less or equal than $n-2$), on $p_m(P)$, and at least on the first point of R. In order to construct an upward drawing of $\mathcal{P}_2$, vertex v_1 has to be mapped on $p_m(P)$ and edge (v_2, v_3), that exists since $n \geq 3$, is a straight-line segment $\overline{q_1 r_1}$. Hence, edge (r, v_1) crosses edge (v_2, v_3). Analogously, r can not be placed on any point of R between the $\frac{n-1}{2}$-th and the $(n-2)$-th.

Now suppose that r is placed in any point of Q between the n-th and the $\frac{3n-3}{2}$-th. Then, by Lemma 6, one out of $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, say $\mathcal{P}^*$, is drawn on a point set S composed of the points of Q above $p(r)$ (that are less or equal than $(n+1)/2$), of $p_m(P)$, and of the last points of R, in number such that $|S| = n$. Whichever is the point on which r is drawn, we have $\mathcal{P}^* = \mathcal{P}_1$, either because all points of S have y-coordinate greater than $p(r)$ (and hence neither v_1 nor w_1 can be drawn on a point of S) or because if a subtree different from $\mathcal{P}_1$ is drawn on S, then no other point of P has y-coordinate greater than the one of $p(r)$ (and hence there is no valid placement for u_1). However, in order to construct an upward drawing of $\mathcal{P}_1$, vertex u_1 has to be mapped on $p_M(P)$ and edge (u_2, u_3), that exists since $n \geq 3$, is a straight-line segment $\overline{q_{(3n-1)/2} r_{(3n-1)/2}}$. Hence, edge (r, u_1) crosses edge (u_2, u_3). Analogously, r can not be placed on any point of R between the $(n-1)$-th and the $\frac{3n-3}{2}$-th.

Finally, suppose that r is placed on $q_{(3n-1)/2}$. Then, one subtree is drawn on the points of Q between the $\frac{n-1}{2}$-th and the $\frac{3n-3}{2}$-th, and one subtree is drawn on the first $(n-3)/2$ points of Q, on $p_m(P)$ and on the first $(n+1)/2$ points of R. The last described subtree must clearly be one out of $\mathcal{P}_2$ and $\mathcal{P}_3$, say $\mathcal{P}_2$. Hence, in order to construct an upward drawing of $\mathcal{P}_2$, v_1 must be placed on $p_m(P)$. Since $n \geq 5$, then $\frac{n-3}{2} \geq 1$, and hence $\mathcal{P}_2$ has an edge (v_2, v_3) drawn as segment $\overline{q_1 r_1}$. It follows that edge (r, v_1) crosses edge (v_2, v_3). Analogously, r can not be placed on $r_{(3n-1)/2}$, and this concludes the proof. $\square$

4 Upward Simultaneous Embeddings of Directed Graphs

In the first part of this section we show that there exist two n-vertex upward planar directed graphs that do not admit an upward simultaneous embedding without mapping.

Let G_n^1 be the graph with n vertices $u_1^1, u_2^1, \cdots, u_n^1$, with a Hamiltonian directed path $\mathcal{P}_1 = (u_1^1, u_2^1, \cdots, u_n^1)$, with edges (u_1^1, u_3^1), (u_1^1, u_4^1), (u_1^1, u_5^1), (u_2^1, u_4^1), and (u_2^1, u_5^1), and with any other set of edges such that G_n^1 is still upward planar. Let G_n^2 be the graph with n vertices $u_1^2, u_2^2, \cdots, u_n^2$, with a Hamiltonian directed path $\mathcal{P}_2 = (u_1^2, u_2^2, \cdots, u_n^2)$, with edges (u_1^2, u_3^2), (u_1^2, u_4^2), (u_1^2, u_5^2), (u_2^2, u_4^2), and (u_3^2, u_5^2), and with any other set of edges such that G_n^2 is still upward planar. We have the following:

Theorem 6 *For every $n \geq 5$, there exist two n-vertex upward planar directed graphs that do not admit a simultaneous embedding without mapping.*

Proof: Suppose that a simultaneous embedding without mapping (Γ_1, Γ_2) of G_n^1 and G_n^2 exists. Since both graphs have Hamiltonian directed paths and since Γ_1 and Γ_2 are supposed to be upward, vertex u_j^1 must be placed at the same point where vertex u_j^2 is placed, for each $1 \leq j \leq n$. It follows that the drawings of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide in (Γ_1, Γ_2). We claim that there exist two edges (u_k^1, u_l^1) and (u_k^2, u_l^2), with $1 \leq k < l \leq 5$, that, in order to respect the planarity of Γ_1 and Γ_2, are drawn one on the left of $\mathcal{P}_1$ and $\mathcal{P}_2$, and the other on their right. Notice that the claim implies a contradiction: since vertex u_k^1 coincides with vertex u_k^2 and vertex u_l^1 coincides with vertex u_l^2, then edges (u_k^1, u_l^1) and (u_k^2, u_l^2) are segments between the same two points. That is, they are coincident and can not be on different sides of $\mathcal{P}_1$ and $\mathcal{P}_2$.

Subgraph G_5^1 (resp. G_5^2) of G_n^1 (resp. of G_n^2) induced by vertices $(u_1^1, u_2^1, \cdots, u_5^1)$ (resp. $(u_1^2, u_2^2, \cdots, u_5^2)$) is triangulated and hence it has only one planar embedding, up to a reversal of the adjacency lists of all vertices. In particular, observe that deciding to place an edge out of (u_1^1, u_3^1), (u_1^1, u_4^1), (u_2^1, u_4^1), and (u_2^1, u_5^1) (resp. of (u_1^2, u_3^2), (u_1^2, u_4^2), (u_2^2, u_4^2), and (u_3^2, u_5^2)) on the left or on the right of $\mathcal{P}_1$ and $\mathcal{P}_2$ determines also on which side of $\mathcal{P}_1$ and $\mathcal{P}_2$ lie the other three edges. This property comes from the fact that, in order to respect the planarity of Γ_1 (resp. of Γ_2), two segments (u_i^1, u_p^1) and (u_l^1, u_q^1) (resp. (u_i^2, u_p^2) and (u_l^2, u_q^2)), with $i < l < p < q$, can not be on the same side of $\mathcal{P}_1$ and $\mathcal{P}_2$. Suppose that edge (u_1^1, u_3^1) of G_n^1 is placed on the left of $\mathcal{P}_1$ and $\mathcal{P}_2$, the case in which it is placed on their right being analogous. Then edges (u_2^1, u_4^1) and (u_2^1, u_5^1) lie on the right of $\mathcal{P}_1$ and $\mathcal{P}_2$, and edge (u_1^1, u_4^1) is on the left of $\mathcal{P}_1$ and $\mathcal{P}_2$. Edge (u_1^2, u_3^2) of G_n^2 is placed on the left of $\mathcal{P}_1$ and $\mathcal{P}_2$, since it coincides with edge (u_1^1, u_3^1) of G_n^1. Then, edge (u_2^2, u_4^2) lies on the right of $\mathcal{P}_1$ and $\mathcal{P}_2$ and edge (u_3^2, u_5^2) is placed on the left. Finally, this implies that edge (u_1^2, u_4^2) is placed on the right of $\mathcal{P}_1$ and $\mathcal{P}_2$, proving the claim and hence the theorem. □

Next, we show that deciding whether two upward planar directed graphs admit a simultaneous embedding without mapping is an $\mathcal{NP}$-hard problem. More formally, the problem is defined as follows:

Problem:	UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING
Instance:	Two n-vertex upward planar directed graphs G_1 and G_2.
Question:	Does a simultaneous embedding without mapping of G_1 and G_2 exist?

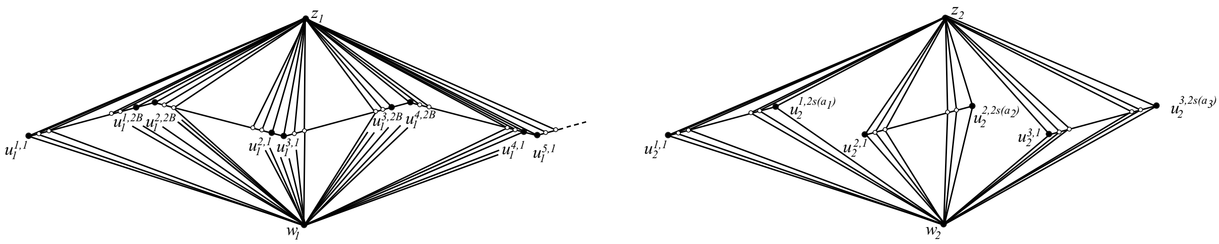
In order to prove the $\mathcal{NP}$-hardness of the previous problem, consider any instance of 3-PARTITION (see, e.g., [9]):

Problem:	3-PARTITION
Instance:	Set A of $3m$ elements, a bound $B \in Z^+$, and a size $s(a) \in Z^+$ for each element $a \in A$ such that $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = mB$.
Question:	Can A be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that, for $1 \leq i \leq 3$ and $1 \leq j \leq m$, $\sum_{a_i \in A_j} s(a_i) = B$?

First, we describe how to construct an instance of UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING from an instance of 3-PARTITION.

Graph G_1 contains m directed paths $\mathcal{P}_i = (u_1^{i,1}, u_1^{i,2}, \dots, u_1^{i,2B})$ of $2B$ vertices, with $1 \leq i \leq m$. Edge $(u_1^{i,j}, u_1^{i,j+1})$ is directed from $u_1^{i,j}$ to $u_1^{i,j+1}$, for $1 \leq j \leq 2B-1$ and $1 \leq i \leq m$. Further, G_1 has an edge directed from vertex $u_1^{i,2B}$ to vertex $u_1^{i+1,2B}$, for each i odd such that $1 \leq i \leq m-1$, and an edge directed from vertex $u_1^{i+1,1}$ to vertex $u_1^{i,1}$, for each i even such that $2 \leq i \leq m$. Finally, G_1 has two vertices w_1 and z_1 such that, for every vertex $u_1^{i,j}$, with $1 \leq i \leq m$ and $1 \leq j \leq 2B$, there exists an edge from $u_1^{i,j}$ to z_1 and an edge from w_1 to $u_1^{i,j}$ (see Fig. 4.a). It is easy to see that G_1 has only one upward planar embedding, up to a flip of the whole graph. Further, the subgraph $\mathcal{P}$ of G_1 induced by the all the vertices of G_1 except for w_1 and z_1 is a directed path. We say that two vertices $u_1^{i_1,j_1}$ and $u_1^{i_2,j_2}$ of G_1 are *consecutive* in G_1 if they are adjacent in $\mathcal{P}$.

Graph G_2 has a triconnected component G_2^i for each element $a_i \in A$ (except for the elements a_i such that $s(a_i) = 1$ to which biconnected components G_2^i correspond). All the G_2^i's share two vertices w_2 and z_2. Graph G_2^i has $2 \cdot s(a_i) + 2$ vertices, where $2 \cdot s(a_i)$ vertices form a directed path $(u_2^{i,1}, u_2^{i,2}, \cdots, u_2^{i,2 \cdot s(a_i)})$ such that edge $(u_2^{i,j}, u_2^{i,j+1})$ is directed from $u_2^{i,j}$ to $u_2^{i,j+1}$, for $1 \leq j \leq 2 \cdot s(a_i) - 1$. For every $1 \leq i \leq 3m$ and $1 \leq j \leq 2 \cdot s(a_i)$, there is an edge directed from $u_2^{i,j}$ to z_2 and an edge directed from w_2 to $u_2^{i,j}$ (see Fig. 4.b). Notice that, an embedding of G_2 is completely specified by a left-to-right order of the G_2^i's, up to flip of some G_2^i's. We say that two vertices $u_2^{i_1,j_1}$ and $u_2^{i_2,j_2}$ of G_2 are *consecutive* in G_2 if $i_1 = i_2$ and $j_2 = j_1 \pm 1$. Notice that, in any upward simultaneous embedding of G_1 and G_2, vertex w_1 of G_1 must be mapped to vertex w_2 of G_2 and vertex z_1 of G_1 must be mapped to vertex z_2 of G_2. We observe the following:



(a) (b)

Figure 4: (a)-(b) Graphs G_1 and G_2. In order to improve readability, labels are shown just for the first and the last vertex of the paths $\mathcal{P}_i$ of G_1 and for the first and the last vertex of the paths of each component G_2^i. An edge (a, b) is oriented from a to b if, in the figure, the y-coordinate of b is greater than the one of a.

Lemma 7 *Let $u_2^{i,j}$ and $u_2^{i,j+1}$ be two consecutive vertices of G_2, for some $1 \leq i \leq 3m$ and $1 \leq j \leq 2 \cdot s(a_i) - 1$. Then, in any upward simultaneous embedding without mapping of G_1 and G_2, vertices $u_2^{i,j}$ and $u_2^{i,j+1}$ are mapped to consecutive vertices of G_1.*

Proof: Suppose, for a contradiction, that there exists an upward simultaneous embedding without mapping (Γ_1, Γ_2) of G_1 and G_2 in which vertices $u_2^{i,j}$ and $u_2^{i,j+1}$ are mapped to two non-consecutive vertices $u_1^{k_1,l_1}$ and $u_1^{k_2,l_2}$ of G_1. Consider the polygon Δ_1 in Γ_1 composed of edges $(w_1, u_1^{k_1,l_1})$, $(w_1, u_1^{k_2,l_2})$, $(z_1, u_1^{k_1,l_1})$, and $(z_1, u_1^{k_2,l_2})$. Polygon Δ_1, by definition of upward simultaneous embedding without mapping, coincides with the polygon in Γ_2 composed of edges $(w_2, u_2^{i,j})$, $(w_2, u_2^{i,j+1})$, $(z_2, u_2^{i,j})$, and $(z_2, u_2^{i,j+1})$. Further, by the uniqueness of the embedding of G_1 and by the fact that $u_1^{k_1,l_1}$ and $u_1^{k_2,l_2}$ are not consecutive in $\mathcal{P}$, Δ_1 contains a vertex $u_1^{x_1,y_1}$ of G_1 in its interior. Consider the vertex $u_2^{x_2,y_2}$ that is mapped to $u_1^{x_1,y_1}$ in (Γ_1, Γ_2). There exist edges directed from $u_2^{x_2,y_2}$ to w_2 and z_2 in G_2. Since such edges are contained inside Δ_1, one of them intersects edge $(u_2^{i,j}, u_2^{i,j+1})$ of G_2, that is also contained inside Δ_1 and that separates w_2 and z_2 inside Δ_1. □

Corollary 1 *Consider any upward simultaneous embedding without mapping of G_1 and G_2 in which two vertices $u_1^{i_1,j_1}$ and $u_1^{i_2,j_2}$ of G_1 have been mapped to two vertices of the same component G_2^k of G_2. Then all vertices between $u_1^{i_1,j_1}$ and $u_1^{i_2,j_2}$ in $\mathcal{P}$ have been mapped to vertices of G_2^k.*

Lemma 8 *In any upward simultaneous embedding without mapping of G_1 and G_2 there exists no component G_2^i of G_2 which has two vertices mapped to vertices $u_1^{j,2B}$ and $u_1^{j+1,2B-1}$, for every j odd, and there exists no component G_2^i of G_2 which has two vertices mapped to vertices $u_1^{j,1}$ and $u_1^{j+1,2}$, for every j even.*

Proof: We prove only the first of the two statements, the other one being analogous. Suppose, for a contradiction, that there exists an upward simultaneous embedding without mapping (Γ_1, Γ_2) of G_1 and G_2 and indices i and j, with j odd, such that a component G_2^i of G_2 has two vertices $u_2^{i,x}$ and $u_2^{i,y}$ mapped to vertices $u_1^{j,2B}$ and $u_1^{j+1,2B-1}$. Consider the polygon Δ_1 in Γ_1 composed of edges $(w_1, u_1^{j,2B})$, $(w_1, u_1^{j+1,2B-1})$, $(z_1, u_1^{j,2B})$, and $(z_1, u_1^{j+1,2B-1})$. Polygon Δ_1, by definition of upward simultaneous embedding without mapping, coincides with the polygon in Γ_2 composed of edges $(w_2, u_2^{i,x})$, $(w_2, u_2^{i,y})$, $(z_2, u_2^{i,x})$, and $(z_2, u_2^{i,y})$. Further, by the uniqueness of the embedding of G_1, Δ_1 contains vertex $u_1^{j+1,2B}$ of G_1 in its interior. No vertex of the directed path $\mathcal{Q}_2(u_2^{i,x}, u_2^{i,y})$ connecting vertices $u_2^{i,x}$ and $u_2^{i,y}$ in G_2^i can be mapped to $u_1^{j+1,2B}$ because the y-coordinate of each vertex of $\mathcal{Q}_2(u_2^{i,x}, u_2^{i,y})$ is smaller than the y-coordinate of the one of $u_2^{i,x}$ and $u_2^{i,y}$ with greater y-coordinate, while y-coordinate of $u_1^{j+1,2B}$ is greater than the ones of $u_1^{j,2B}$ and $u_1^{j+1,2B-1}$. Hence, path $\mathcal{Q}_2(u_2^{i,x}, u_2^{i,y})$ crosses the edge of G_2 from w_2 to the vertex of G_2 that is mapped to $u_1^{j+1,2B}$. □

We obtain the following:

Theorem 7 *UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING is $\mathcal{NP}$-hard.*

Proof: The reduction described at the beginning of the section can clearly be performed in polynomial time. In fact, 3-PARTITION is $\mathcal{NP}$-hard in the strong sense, hence it is $\mathcal{NP}$-hard even if $2mB$, which is the size of the constructed instance of UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING, is bounded by a polynomial in m. We show that an instance of 3-PARTITION admits a solution if and only if the corresponding instance of UPWARD SIMULTANEOUS EMBEDDING WITHOUT MAPPING admits a solution.

Consider any instance A of 3-PARTITION admitting a solution $A_1, A_2, \dots, A_m$ such that, for $1 \leq i \leq 3$ and $1 \leq j \leq m$, $\sum_{a_i \in A_j} s(a_i) = B$. We show how to map the vertices of G_2 to the vertices of G_1 and then how to construct an upward simultaneous embedding of the two graphs. For each $i = 1, \cdots, m$, consider the three components G_2^x, G_2^y, and G_2^z of G_2 corresponding to the three elements a_x, a_y, and a_z of A_i, respectively. Notice that, since G_2^k has $2 \cdot s(a_k)$ vertices different from w_2 and z_2, then G_2^x, G_2^y, and G_2^z have exactly $2B$ vertices different from w_2 and z_2. Map vertex $u_2^{x,j}$ of G_2^x to vertex $u_1^{i,j}$ of G_1, for each $1 \leq j \leq 2 \cdot s(a_x)$; map vertex $u_2^{y,j}$ of G_2^y to vertex $u_1^{i,2 \cdot s(a_x)+j}$ of G_1, for each $1 \leq j \leq 2 \cdot s(a_y)$; map vertex $u_2^{z,j}$ of

G_2^z to vertex $u_1^{i,2\cdot s(a_x)+2\cdot s(a_y)+j}$ of G_1, for each $1 \leq j \leq 2 \cdot s(a_z)$. It is easy to see that G_2, when its vertices are mapped to the vertices of G_1 as described above, is a subgraph of G_1. Hence, an upward simultaneous embedding of G_1 and G_2 is obtained by any upward straight-line drawing of G_1.

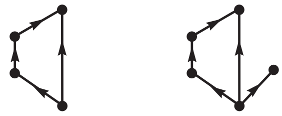
Now consider any upward simultaneous embedding (Γ_1, Γ_2) of G_1 and G_2. We show how to construct a solution $A_1, A_2, \ldots, A_m$ for the corresponding instance A of 3-PARTITION. We claim that the vertices of three components $G_2^{x,i}$, $G_2^{y,i}$, and $G_2^{z,i}$ of G_2, except for w_2 and z_2, have been mapped to all and only the vertices of path $\mathcal{P}_i$ of G_1, for each $i = 1, 2, \cdots, m$. The claim directly implies the existence of a solution to the instance of 3-PARTITION, since the claim implies that $|G_2^{x,i}| + |G_2^{y,i}| + |G_2^{z,i}| = 2B$ and hence, by construction, the three elements of A corresponding to $G_2^{x,i}$, $G_2^{y,i}$, and $G_2^{z,i}$ sum up to B. Consider the component $G_2^{x,1}$ of G_2 which has a vertex mapped to $u_1^{1,1}$. By Corollary 1, the vertices of $G_2^{x,1}$ are mapped to consecutive vertices of G_1, hence they are mapped to the vertices of $\mathcal{P}$ from $u_1^{1,1}$ to $u_1^{1,|G_2^{x,1}|}$. Analogously, the component $G_2^{y,1}$ of G_2 which has a vertex mapped to $u_1^{1,|G_2^{x,1}|+1}$ has vertices mapped to vertices of $\mathcal{P}$ from $u_1^{1,|G_2^{x,1}|+1}$ to $u_1^{1,|G_2^{x,1}|+|G_2^{y,1}|}$. Observe that $|G_2^{x,1}|+|G_2^{y,1}| < 2B$, since each component of G_2 has less than B vertices (by the assumption that $s(a_i) < B/2$). Hence, there exists a component $G_2^{z,1}$ of G_2 which has a vertex mapped to $u_1^{1,|G_2^{x,1}|+|G_2^{y,1}|+1}$. By Corollary 1, the vertices of $G_2^{z,1}$ are mapped to consecutive vertices of G_1. Suppose that $|G_2^{x,1}|+|G_2^{y,1}|+|G_2^{z,1}| > 2B$. Since each of $|G_2^{x,1}|$, $|G_2^{y,1}|$, and $|G_2^{z,1}|$ is even, $|G_2^{x,1}|+|G_2^{y,1}|+|G_2^{z,1}|$ is even, as well, hence $G_2^{z,1}$ has a vertex mapped to $u_1^{2,2B-1}$. By Lemma 8, (Γ_1, Γ_2) is not an upward simultaneous embedding without mapping of G_1 and G_2. Now suppose that $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| < 2B$. Then, there exists a component $G_2^{w,1}$ that is mapped to $u_1^{1,|G_2^{x,1}|+|G_2^{y,1}|+|G_2^{z,1}|+1}$. By Corollary 1, the vertices of $G_2^{w,1}$ are mapped to consecutive vertices of G_1. Further, by construction and by the assumption that $s(a_i) > B/4$, $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| + |G_2^{w,1}| > 2B$. Hence, since $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| + |G_2^{w,1}|$ is even, $G_2^{w,1}$ has vertices mapped both to $u_1^{1,2B}$ and to $u_1^{2,2B-1}$. By Lemma 8, (Γ_1, Γ_2) is not an upward simultaneous embedding without mapping of G_1 and G_2. It follows that $|G_2^{x,1}| + |G_2^{y,1}| + |G_2^{z,1}| = 2B$. The previous argument can be iterated to show that the vertices of three components $G_2^{x,i}$, $G_2^{y,i}$, and $G_2^{z,i}$ of G_2, except for w_2 and z_2, have been mapped to all and only the vertices of path $\mathcal{P}_i$ of G_1, for each $i = 1, 2, \cdots, m$, proving the claim and hence the theorem. □

5 Conclusions and Open Problems

In this paper we have shown combinatorial and complexity results regarding the problem of constructing upward straight-line embeddings of directed graphs into point sets.

We have shown families of directed graphs that admit a straight-line upward embedding into every point set in convex position or in general position, and families that do not. However, the problem of characterizing those graphs admitting a straight-line upward embedding into every point set in general or in convex position is still open. In such a direction, we remark that if an upward planar directed graph admits an upward straight-line embedding into every point set in general or in convex position, not all its subgraphs, in general, admit an upward straight-line embedding into every point set in general or in convex position; see Fig. 5. However, the necessary conditions of Lemmata 2, 3, and 4 strongly restrict the class of upward planar directed graphs to investigate.

We proved that deciding whether two upward planar directed graphs admit an upward si-



(a) (b)

Figure 5: (a) An upward planar directed graph G_1 that, by Theorem 1, does not admit an upward straight-line embedding into every 4-point point set in convex position. (b) An upward planar directed graph G_2 that admits an upward straight-line embedding into every 5-point point set in convex position and that contains G_1 as a subgraph.

multaneous embedding without mapping is $\mathcal{NP}$-hard. We observe that the same problem is polynomial-time solvable for trees [10]. Hence, it would be interesting to solve the problem for subclasses of planar digraphs richer than directed trees, e.g., *outerplanar digraphs* and *series-parallel digraphs*.

Finally, the problem of determining the minimum cardinality $f(n)$ of a set of points P in the plane such that every n-vertex planar directed graph admits an upward straight-line embedding in which the vertices are drawn at points of P, has not been deeply investigated, even if it is the directed version of one of the most studied Graph Drawing problems [5, 6, 13, 14]. We remark that here any polynomial upper bound for $f(n)$ would be interesting, since the only known result concerning the problem is that the minimum size of any grid into which every planar directed graph can be drawn is exponential [8].

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