Università degli Studi di Roma Tre
Dipartimento di Informatica e Automazione
Via della Vasca Navale, 79 – 00146 Roma, Italy

C-Planarity of C-Connected Clustered Graphs

Part I – Characterization

P. F. Cortese, G. Di Battista, F. Frati, M. Patrignani, and M. Pizzonia

RT-DIA-109-2006 **June 2006**

Dipartimento di Informatica e Automazione,
Università di Roma Tre,
Rome, Italy.
{cortese,gdb,frati,patrigna,pizzonia}@dia.uniroma3.it

Work partially supported by European Commission - Fet Open project DELIS - Dynamically Evolving Large Scale Information Systems - Contract no 001907 and by "Project ALGO-NEXT: Algorithms for the Next Generation Internet and Web: Methodologies, Design, and Experiments", MIUR Programmi di Ricerca Scientifica di Rilevante Interesse Nazionale.

ABSTRACT

We present a characterization of the c-planarity of c-connected clustered graphs. The characterization is based on the interplay between the hierarchy of the clusters and the hierarchy of the triconnected and biconnected components of the graph underlying the clustered graph. In a companion paper [2] we exploit such a characterization to give a linear time c-planarity testing and embedding algorithm.

1 Introduction

In most networks it is possible to find semantic relationships among components that allow to group them into clusters. For example, in a social network representing employees of a company and their work relationships it could be desirable to group the people of each department together. As another example, the routers of the Internet are often grouped into areas, that in turn can be grouped into Autonomous Systems.

When representing a network containing clusters, it is quite common to draw the elements of each cluster inside the same region of the plane. Also, disjoint clusters tipically lie into disjoint regions. The clustered planarity field studies the interplay between the classical planarity of graphs and the presence of clusters. For its practical interest and because of its theoretical appeal, clustered planarity is attracting increasing attention.

A network containing clusters is called clustered graph. More formally, a *clustered graph* $C(G,T)$ consists of a graph G and of a rooted tree T such that the leaves of T are the vertices of G (see Fig. 1.a and 1.b). Each internal node ν of T corresponds to the subset $V(\nu)$ of the vertices of G (called *cluster*) that are the leaves of the subtree rooted at ν. Each non-leaf node of T has at least two children. The subgraph of G induced by $V(\nu)$ is denoted as $G(\nu)$. An edge e between a vertex of $V(\nu)$ and a vertex of $V - V(\nu)$ is said to be *incident* on ν. Graph G and tree T are called *underlying graph* and *inclusion tree*, respectively. A clustered graph is *c-connected* if for each node ν of T we have that $G(\nu)$ is connected (e.g., the clustered graph in Fig. 1.a and 1.b is c-connected).

In a *drawing* of a clustered graph $C(G,T)$ vertices and edges of G are drawn as points and curves [4], and each node ν of T is a simple closed region $R(\nu)$ such that: (i) $R(\nu)$ contains the drawing of $G(\nu)$; and (ii) $R(\nu)$ contains a region $R(\mu)$ if and only if μ is a descendant of ν in T (see Fig. 1.a). Consider an edge e and a node ν of T. If e crosses the boundary of $R(\nu)$ more than once, we say that edge e and region $R(\nu)$ have an *edge-region crossing*. A drawing of a clustered graph is *c-planar* if it does not have edge crossings and edge-region crossings. A clustered graph is *c-planar* if it admits a c-planar drawing (e.g., the clustered graph in Fig. 1.a and 1.b is c-planar).

The problem of characterizing the c-planar clustered graphs has been tackled from many points of view and from many authors. Namely, Feng, Cohen, and Eades [7] have shown that a clustered graph $C(G,T)$ is c-planar iff G has a planar embedding such that for each node ν of T all the vertices and edges of $G - G(\nu)$ are on the external face of the embedding of $G(\nu)$. However, such a characterization is more a characterization of the c-planar embeddings rather than a structural characterization of the c-planarity. A characterization of the c-planar embeddings has been given also by Dahlhaus in [3]. A further elegant characterization has been given by Cornelsen and Wagner [1] for a subclass of clustered graphs. Namely, a completely connected clustered graph is c-planar iff its underlying graph is planar, where completely connected means that for each node ν of T $G(\nu)$ and $G - G(\nu)$ are connected (e.g., the clustered graph in Fig. 1.a and 1.b is not completely connected).

In this paper we present the first (as far as we know) structural characterization of the c-planarity of c-connected clustered graphs whose underlying graph is biconnected. The characterization is based on the interplay between the hierarchy T of the clusters and the hierarchy of the triconnected components of the underlying graph G. It is given in terms of properties of the skeletons of the nodes of the SPQR-tree of G. Notice that in at least two other papers [3, 8] the relationship between triconnectivity and c-planarity has been already studied. We also easily extend our characterization to general clustered graphs making use of the decomposition of G into its biconnected components. In a companion paper [2] we exploit the characterization of this paper to give a linear time testing and embedding algorithm for the c-planarity of c-connected clustered graphs.

The paper is organized as follows. In Section 2 we give basic definitions. In Section 3 we present our characterization. In Sections 4 and 5 we prove the correctness. Section 6 extends the result to general clustered graphs.

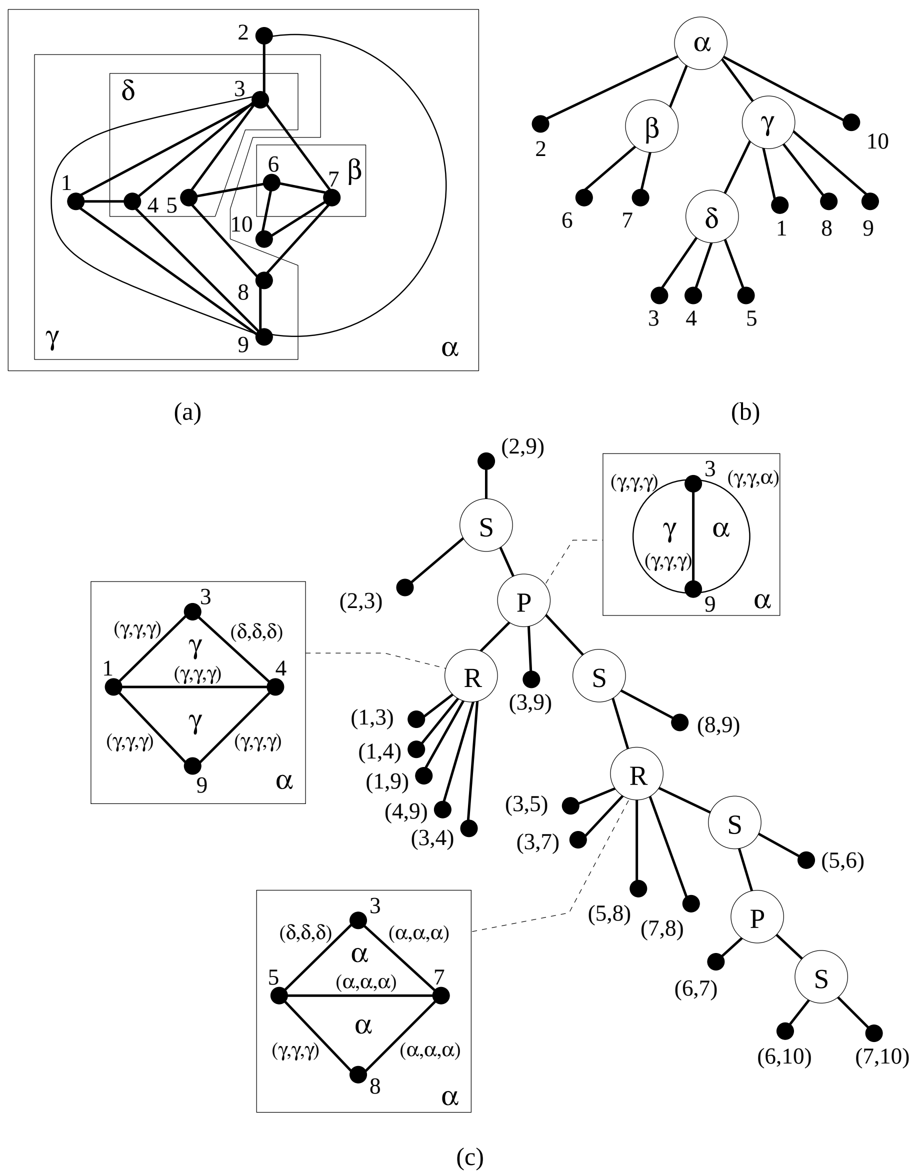


Figure 1: Clustered graph. (a) Underlying graph. (b) Inclusion tree. (c) SPQR-tree. The boxes contain the skeletons of selected nodes. The triple on each virtual edge represents $lcc(e)$, $lsc(e)$, and $hsc(e)$, respectively. Faces of skeletons are labelled with their lcc.

2 Background

We assume familiarity with planarity and connectivity of graphs [6]. We also assume some familiarity with graph drawing [4].

A *drawing* D of a graph $G(V, E)$ is a mapping of each vertex $v \in V$ to a distinct point $D(v)$ of the plane and of each edge $(u, v) \in E$ to a Jordan curve joining points $D(u)$ and $D(v)$. Given a drawing D of a graph G, two edges *intersect* if they share a point which is not a common end-vertex. A drawing with no intersection is *planar* and a graph is *planar* if it admits a planar drawing. A planar drawing partitions the plane into topologically connected regions called *faces*. The unbounded face is called the *external face*. Let G be a planar graph. An *embedded planar graph* G_Γ is an equivalence class of planar drawings of G with the same circular ordering for the adjacency lists of vertices and with the same external face. Such a choice Γ of a circular ordering of the adjacency lists and of the external face is called *planar embedding* of G.

A graph $G(V, E)$ is *connected* if every pair of vertices of G is connected by a path. A *separating k-set* of a graph G is a set of k vertices whose removal increases the number of connected components of G. Separating 1-sets and 2-sets are called *cutvertices* and *separation pairs*, respectively. A connected graph is said to be *biconnected* if it has no cutvertices. The maximal biconnected subgraphs of a graph are its *blocks*. Observe that each edge of G falls into a single block of G, while cutvertices are shared by different blocks. The *block cutvertex tree*, or BC-tree, of a connected graph G has a B-node for each block of G and a C-node for each cutvertex of G. Edges in the BC-tree connect each B-node μ to the C-nodes associated with the cutvertices in the block of μ. The BC-tree of G may be thought as rooted at a specific block ν. Consider a cutvertex v of a BC-tree rooted at ν. The graph obtained by merging the blocks of which v is an ancestor is called the *pertinent graph* of v, and is denoted by $\text{pertinent}(v)$.

A *split pair* $\{u, v\}$ of a graph G is either a separation pair or a pair of adjacent vertices. A *maximal split component* of G with respect to a split pair $\{u, v\}$ (or, simpler, a maximal split component of $\{u, v\}$) is either an edge (u, v) or a maximal subgraph G' of G such that G' contains u and v and $\{u, v\}$ is not a split pair of G'. A vertex w distinct from u and v belongs to exactly one maximal split component of $\{u, v\}$. We call the *split component* of $\{u, v\}$ a subgraph of G that is the union of any number of maximal split components of $\{u, v\}$.

In the following, we summarize SPQR-trees. For more details, see [5]. SPQR-trees are closely related to the classical decomposition of biconnected graphs into triconnected components. Let $\{s, t\}$ be a split pair of G. A *maximal split pair* $\{u, v\}$ of G with respect to $\{s, t\}$ is a split pair of G distinct from $\{s, t\}$ such that, for any other split pair $\{u', v'\}$ of G, there exists a split component of $\{u', v'\}$ containing vertices u, v, s, and t. Let $e = (s, t)$ be an edge of G, called *reference edge*. The SPQR-tree $\mathcal{T}$ of G with respect to e describes a recursive decomposition of G induced by its split pairs. Tree $\mathcal{T}$ is a rooted ordered tree whose nodes are of four types: S, P, Q, and R. Denote by G' the st-biconnectible graph obtained from G by removing e. Each node μ of $\mathcal{T}$ has an associated st-biconnectible multigraph, called the *skeleton* of μ and denoted by $\text{skeleton}(\mu)$. Also, it is associated with an edge of the skeleton of the parent ν of μ, called the *virtual edge* of μ in *skeleton*(ν). Tree $\mathcal{T}$ is recursively defined as follows.

Trivial Case: If G consists of exactly one edge between s and t, then $\mathcal{T}$ consists of a single Q-node whose skeleton is G itself.

Parallel Case: If the split pair $\{s, t\}$ has at least two maximal split components $G_1, \ldots, G_k$ ($k \geq 2$), the root of $\mathcal{T}$ is a P-node μ. Graph *skeleton*(μ) consists of k parallel edges between s and t, denoted $e_1, \ldots, e_k$.

Series Case: If the split pair $\{s, t\}$ has exactly one maximal split component G' which is not a single edge and if G' has cutvertices $c_1, \ldots, c_{k-1}$ ($k \geq 2$) in this order on a path from s to t, the root of $\mathcal{T}$ is an S-node μ. Graph *skeleton*(μ) is the path $e_1, \ldots, e_k$, where e_i connects c_{i-1} with c_i ($i = 2 \ldots k-1$), e_1 connects s with c_1, and e_k connects c_{k-1} with t.

Rigid Case: If none of the above cases applies, let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be the maximal split pairs of G with respect to $\{s, t\}$ ($k \geq 1$) and, for $i = 1, \ldots, k$, let G_i be the union of all the maximal split components of $\{s_i, t_i\}$. The root of $\mathcal{T}$ is an R-node μ. Graph *skeleton*(μ) is the triconnected graph obtained from G by replacing each subgraph G_i with the edge e_i between s_i and t_i.

Except for the trivial case, μ has children $\mu_1, \ldots, \mu_k$, in this order, such that μ_i is the root of the SPQR-tree of graph $G_i \cup (u_i, v_i)$ with respect to reference edge (u_i, v_i) ($i = 1, \ldots, k$). Edge (u_i, v_i) is said to be the *virtual edge* of node μ_i in *skeleton*(μ) and of node μ in *skeleton*(μ_i). Graph G_i is called the *pertinent graph* of node μ_i, and of edge (u_i, v_i) and it is denoted by pertinent(u_i, v_i). Vertices u and v are the *poles* of G_i.

The tree $\mathcal{T}$ so obtained has a Q-node associated with each edge of G, except the reference edge e. We complete the SPQR-tree $\mathcal{T}$ by adding another Q-node, representing the reference edge e, and making it the parent of μ so that it becomes the root of $\mathcal{T}$ (see Fig. 1.c for an example).

The SPQR-tree $\mathcal{T}$ of a graph G with n vertices and m edges has m Q-nodes and $O(n)$ S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of $\mathcal{T}$ is $O(n)$.

A graph G is planar if and only if the skeletons of all the nodes of the SPQR-tree $\mathcal{T}$ of G are planar. An SPQR-tree $\mathcal{T}$ rooted at a given Q-node can be used to represent all the planar embeddings of G having the reference edge (associated with the Q-node at the root) on the external face. Namely, any embedding can be obtained by selecting one of the two possible flips of each skeleton around its poles and selecting a permutation of the children of each P-node with respect to their common poles.

3 A Characterization of the C-Connected C-Planar Clustered Graphs whose Underlying Graph is Biconnected

In this section we characterize the c-planarity of a c-connected clustered graph $C(G, T)$ when G is planar and biconnected. To do that we need some definitions on the cluster hierarchy. Given a connected subgraph G' of G, the *allocation cluster* of G', denoted by $ac(G')$ is the lowest common ancestor in T of the vertices of G'. The allocation cluster represents the lowest cluster containing all the vertices of G'. In the following we refer to the allocation clusters of many special subgraphs of G like edges, paths, and cycles. For example in Fig. 1 the allocation cluster of path $3, 5, 8$ is γ. Two clusters α and β of T

are *comparable* when they are on the same path from a leaf to the root of T. If α and β are comparable, the operators $\prec$, $\preceq$, and max are defined, where $\alpha \preceq \beta$ ($\alpha \prec \beta$) means that α is an ancestor (proper ancestor) of β and $\max(\alpha, \beta)$ is the farthest cluster from the root. The following properties are easily proved.

Property 1 *Given two connected subgraphs G' and G'' of G sharing a vertex, $ac(G')$ and $ac(G'')$ are comparable. Also, if $G'' \subseteq G'$ then $ac(G') \preceq ac(G'')$.*

Property 2 *Let G' be a connected subgraph of G. There is at least one edge $e \in G'$ such that $ac(e) = ac(G')$.*

For example, in Fig. 1 the subgraph induced by vertices $3, 5, 8, 7$ has allocation cluster equal to α and $ac((3, 7)) = \alpha$.

Property 3 *There is at least one edge $e \in G$ such that $ac(e)$ is the root of T.*

Now we relate the concept of allocation cluster, typical of the clusters hierarchy, to the hierarchy of the triconnected components of G, represented by the SPQR-tree $\mathcal{T}$. A *lowest connecting path* of a virtual edge $e = (u, v)$ of the skeleton of a node of $\mathcal{T}$ is a path between u and v in *pertinent*(e) with maximum allocation cluster. Observe that by Property 1 all the paths connecting u and v are pairwise comparable. The *lowest connecting cluster* of e, denoted by $lcc(e)$ is the allocation cluster of the lowest connecting path of e. Fig. 2 shows an example of a virtual edge e and its lowest connecting path. The allocation cluster of path $p_1 = (u, x, w, y, v)$ is β, while the allocation cluster of path $p_2 = (u, x, z, y, v)$ is α. Only p_1 is a lowest connecting path of e, and $lcc(e) = \beta$. Observe that if $e \in G$ the lowest connecting path of e is the edge itself.

Consider a skeleton of a node μ of $\mathcal{T}$ and a path p composed by virtual edges of the skeleton. The *lowest connecting cluster* of p is the lowest common ancestor of the lowest connecting clusters of the edges of p. To have some intuition on this definition, consider that each edge of p corresponds to a pertinent graph that has a lowest connecting path. Hence, we can see p as "representative" of the concatenation of the lowest connecting paths traversing such pertinent graphs. Each of such paths has an allocation cluster; the lowest connecting cluster is the lowest common ancestor of such allocation clusters.

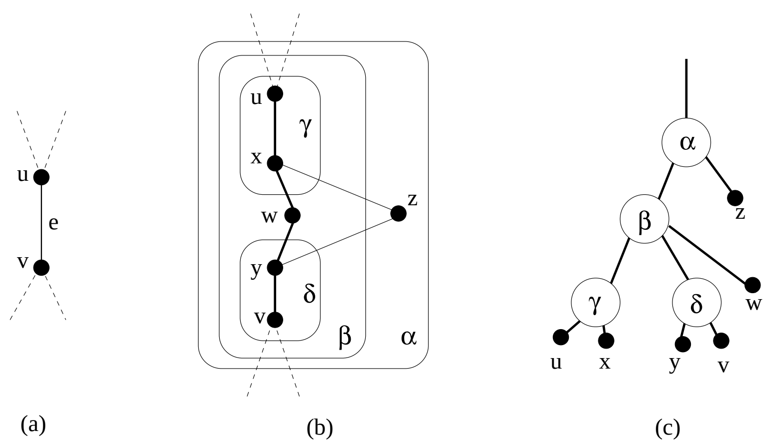


Figure 2: Lowest connecting paths and clusters. (a) A virtual edge $e = (u, v)$. (b) Graph *pertinent*(e); the thick lines show the lowest connecting path p_1. (c) Tree T restricted to the clusters in *pertinent*(e).

We shall adopt the same definition of lowest connecting cluster also for cycles and faces of $skeleton(\mu)$. Also, for technical reasons we define the lowest connecting cluster of an external face as the root of the inclusion tree T. In Fig. 1.c the faces of the skeletons are labelled with the corresponding lowest connecting clusters.

Now we relate the above definitions to the c-planar embeddings. In the following when we refer to a planar embedding of a pertinent graph we always suppose that it has its poles on the external face.

Theorem 1 *A planar embedding of a c-connected clustered graph is c-planar iff it does not exist a cycle c that encloses an edge e such that $ac(e) \prec ac(c)$.*

Proof. Let $C(G,T)$ be a c-connected clustered graph. First, we prove that if a planar embedding Γ of G is c-planar then every cycle c of Γ does not enclose any edge e such that $ac(e) \prec ac(c)$. Suppose that there exist in Γ a cycle c and an edge e such that e is enclosed in c and that $ac(e) \prec ac(c)$. It follows that the region $R(\alpha)$ representing the allocation cluster α of c encloses an edge whose allocation cluster β is a proper ancestor of α. Since $G(\alpha)$ is connected and since Γ is planar, then $R(\beta)$ is enclosed in $R(\alpha)$ and so Γ is not c-planar.

Now by supposing that Γ is a planar embedding of G such that every cycle c of Γ does not enclose any edge e with $ac(e) \prec ac(c)$ we prove that Γ is c-planar. Construct any planar drawing of G with embedding Γ and draw any region $R(\gamma)$ representing a cluster γ of T so that $R(\gamma)$ "surrounds" $G(\gamma)$. The obtained drawing D of C is a c-planar drawing. Namely, since D is a planar drawing then it has no edge crossings; further, the construction of the regions and the c-connectivity of C implies that D has no edge-region crossings. It remains to show that D does not contain two regions R_1 and R_2, respectively associated to clusters α and β of T, such that: (i) R_1 encloses R_2 and (ii) α is not an ancestor of β. This is done by showing that if there exist in D regions R_1 and R_2 that verify conditions (i) and (ii), then there exists a cycle c of Γ that encloses an edge e such that $ac(e) \prec ac(c)$.

By the construction of the regions, there exists in D a cycle c belonging to $G(\alpha)$ that encloses the subgraph $G^* = G(\beta)$. If β is a proper ancestor of α then trivially $ac(e) \prec ac(c)$, for any edge $e \in G^*$. Otherwise, consider the subgraph G' of G that is inside c in D, where $c \in G'$. By the c-connectivity of C, G' is connected. Since α is not an ancestor of β and since β is not a proper ancestor of α then α and β are not comparable. This implies that the allocation cluster of G' is a proper ancestor of both c and G^*, that is $ac(G') \prec ac(c)$ and $ac(G') \prec ac(G^*)$. By Property 2 there exists at least one edge $e \in G'$ such that $ac(e) = ac(G')$. Edge e can not be part of c and so it is enclosed in c. Hence, we obtain $ac(e) = ac(G') \prec ac(c)$, so we derived a contradiction and this concludes the proof. □

An embedding of $skeleton(\mu)$ (μ is a node of $\mathcal{T}$) is *c-planar* when any cycle c of $skeleton(\mu)$ does not enclose an edge e of $skeleton(\mu)$ with $lcc(e) \prec lcc(c)$. Intuitively, the c-planarity of a skeleton is the one of the embedded graph obtained by substituting each edge of the skeleton with its lowest connecting path.

Lemma 1 *If an embedding Γ of $skeleton(\mu)$ is c-planar then for any cycle c in Γ and for any face f inside c we have that $lcc(c) \preceq lcc(f)$.*

Proof sketch. Analogous to the one of Theorem 1. □

Given a virtual edge $e = (u, v)$ and a c-planar embedding Γ of $pertinent(e)$, a lowest connecting path s of e separates $pertinent(e)$ into two embedded subgraphs each containing s. We call *highest side* $hs(\Gamma, s)$ and *lowest side* $ls(\Gamma, s)$ such subgraphs, where $ac(hs(\Gamma, s)) \preceq ac(ls(\Gamma, s))$. By Property 1 we have:

Property 4 $ac(hs(\Gamma, s)) = ac(pertinent(e))$.

Hence, the value of $ac(hs(\Gamma, s))$ does not depend on the choice of the c-planar embedding Γ and of s and we can define the *highest side cluster* of e, $hsc(e) = ac(pertinent(e))$.

Lemma 2 *The value of $ac(ls(\Gamma, s))$ does not depend on the choice of s.*

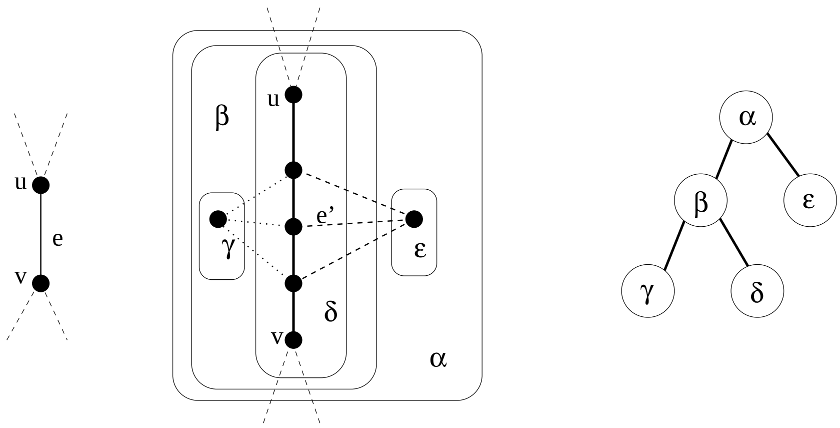
Proof sketch. The proof is based on the fact that if two choices of s give two different values of $ac(ls(\Gamma, s))$, then they form a cycle containing an edge whose allocation cluster is a proper ancestor of the allocation cluster of the cycle. Hence, by Theorem 1 we would have a non c-planarity. $\square$

Due to Lemma 2 we can define the *lowest side cluster* of Γ $lsc(\Gamma) = ac(ls(\Gamma, s))$ and the *lowest side cluster* of e, $lsc(e) = \max_\Gamma\{lsc(\Gamma)\}$. Observe that the definitions of $hsc(e)$ and of $lsc(e)$ hold only if $pertinent(e)$ is c-planar. For technical reasons if $pertinent(e)$ is not c-planar we define $hsc(e) = lsc(e) = \perp$, where $\perp$ is by convention a proper ancestor of any cluster. See Fig. 3 for an example. As another example, Fig. 1 contains, for each virtual edge e of the represented skeletons, a triple describing $lcc(e)$, $lsc(e)$, and $hsc(e)$, respectively.

Property 5 *For each edge e of the skeleton of a node of $\mathcal{T}$, $hsc(e) \preceq lsc(e) \preceq lcc(e)$.*

Property 6 *Let c be a cycle of virtual edges in a skeleton of a node of $\mathcal{T}$ and let e be an edge of c. We have that $lcc(c)$ is comparable with $lcc(e)$, with $lsc(e)$, and with $hsc(e)$.*

Property 7 *Let $e = (u, v)$ be a virtual edge. Suppose that $pertinent(e)$ is c-planar. Then in any c-planar embedding Γ of $pertinent(e)$ and for any lowest connecting path s of e there exist two edges $e_1 \in hs(\Gamma, s)$ and $e_2 \in ls(\Gamma, s)$ such that $ac(e_1) = hsc(e)$ and $ac(e_2) \preceq lsc(e)$.*



(a) (b) (c)

Figure 3: (a) A virtual edge $e = (u, v)$. (b) Graph $pertinent(e)$. (c) Tree T restricted to the clusters in $pertinent(e)$. We have that $lcc(e) = \delta$, $lsc(e) = \beta$, and $hsc(e) = \alpha$. If the edge e' is removed from $pertinent(e)$ then $lsc(e)$ becomes δ.

Two comparable virtual edges e_1 and e_2 of a skeleton of a node of $\mathcal{T}$ are *incompatible* when, assuming w.l.o.g. $lcc(e_1) \preceq lcc(e_2)$, one of the following conditions hold: (i) $lcc(e_1) \prec lcc(e_2)$ and $hsc(e_2) \prec llc(e_1)$; (ii) $lcc(e_1) = lcc(e_2)$, $hsc(e_1) \prec lcc(e_1)$, and $hsc(e_2) \prec llc(e_2)$.

For example, the skeleton of the P-node shown in Fig. 1.c has three virtual edges that are pairwise compatible. Now we can formulate the characterization.

Theorem 2 *Let $C(G,T)$ be a c-connected clustered graph where G is planar and biconnected, and let $\mathcal{T}$ be the SPQR tree of G rooted at an edge whose allocation cluster is the root of T. C is c-planar if and only if for each node μ of $\mathcal{T}$ the following conditions are true:*

1. *If μ is an R node then the embedding of skeleton(μ) is c-planar and each edge e of skeleton(μ) is incident to two faces f_1 and f_2 such that the lowest connecting cluster of f_1 is an ancestor of the highest side cluster of e and the lowest connecting cluster of f_2 is an ancestor of the lowest side cluster of e.*

2. *If μ is a P node then*
   - *(a) it does not exist a set of three edges of skeleton(μ) that are pairwise incompatible and*
   - *(b) there exists at most one edge e^* of skeleton(μ) such that the lowest side cluster of e^* is a proper ancestor of the lowest connecting cluster of e^* and if there exists such e^* then for each edge $e \neq e^*$ of skeleton(μ) the lowest connecting cluster of e is an ancestor of the lowest side cluster of e^*.*

4 Proof of the Sufficiency

We now prove the sufficiency of the conditions of Theorem 2 by structural induction starting from the leaves of $\mathcal{T}$. For each leaf μ (Q node) of $\mathcal{T}$, *pertinent*(μ) is trivially c-planar. We now prove the inductive step by considering any non-leaf node μ with children $\mu_1, \mu_2, \dots, \mu_k$. Namely, we suppose that their pertinent graphs are c-planar and argument that, if the conditions of the theorem are satisfied, then also *pertinent*(μ) is c-planar. This is done by means of two lemmas, one for the R nodes and the other for the P nodes. The lemma for the R nodes is valid for all nodes whose skeleton has a fixed embedding and so it is formulated in the most general setting. The case of the S nodes is trivial. A special lemma describes the situation of the root of $\mathcal{T}$ and completes the proof of sufficiency.

Lemma 3 *Let μ be a node of $\mathcal{T}$ and let Γ_μ be a c-planar embedding of skeleton(μ). If (i) the pertinent graphs of the children of μ are c-planar, and (ii) each edge e of skeleton(μ) is incident to two faces f_1 and f_2 of Γ_μ such that $lcc(f_1) \preceq hsc(e)$ and $lcc(f_2) \preceq lsc(e)$, then pertinent(μ) is c-planar.*

Proof. Let $e_1, e_2, \dots, e_k$ be the edges of *skeleton*(μ) and consider for each e_i, $i = 1, \dots, k$, a c-planar embedding Γ_i of *pertinent*(e_i) such that $lsc(\Gamma_i) = lsc(e_i)$.

Denoting by f_{i1} and f_{i2} the faces of Γ_μ incident to e_i, construct an embedding Γ of *pertinent*(μ) by turning each Γ_i, $i = 1, \dots, k$, so that $hs(\Gamma_i, s_i)$ is toward f_{i1} and $ls(\Gamma_i, s_i)$

is toward f_{i2} with $lcc(f_{i1}) \preceq hsc(e_i)$ and $lcc(f_{i2}) \leq lsc(e)$. The second condition of the lemma makes this possible.

By supposing that Γ_μ is c-planar, we show that Γ is also c-planar, that is for any cycle c of Γ and any edge e that is enclosed in c we have $ac(c) \preceq ac(e)$ (see Theorem 1).

We denote by $e^* = (v_1, v_2)$ the edge of $skeleton(\mu)$ containing e in $pertinent(e^*)$ and we denote by f_1^* and f_2^* the faces incident to e^* in Γ_μ for which $lcc(f_1^*) \preceq hsc(e^*)$ and $lcc(f_2^*) \preceq lsc(e^*)$. Let Γ^* be the embedding of $pertinent(e^*)$ in Γ. We also denote by c^* the cycle in Γ_μ whose edges e_i contain in $\bigcup(pertinent(e_i))$ all the edges of c.

Edge e^* may be part of c^* or can be enclosed in c^*. In both cases at least one of f_1^* and f_2^* is enclosed by c^*. If c^* encloses f_1^*, we have $hsc(e^*) \preceq ac(e)$ by definition of $hsc(e^*)$, $lcc(f_1^*) \preceq hsc(e^*)$ by hypothesis and by the construction of Γ, $lcc(c^*) \preceq lcc(f_1^*)$ by Lemma 1, and $ac(c) \preceq lcc(c^*)$ by definition of lowest conncting cluster. Hence, $ac(c) \preceq lcc(c^*) \preceq lcc(f_1^*) \preceq hsc(e^*) \preceq ac(e)$.

If c^* encloses f_2^* (and does not enclose f_1^*), consider a lowest connecting path s of e^* and the path $p \subset c$ between v_1 and v_2 in $pertinent(e^*)$. If $e \in ls(\Gamma^*, s)$ then $lsc(e^*) \preceq ac(e)$ by definition of $lsc(e^*)$, $lcc(f_2^*) \preceq lsc(e^*)$ by hypothesis and by the construction of Γ, $lcc(c^*) \preceq lcc(f_2^*)$ by Lemma 1, and $ac(c) \preceq lcc(c^*)$ by definition of lowest connecting cluster, so $ac(c) \preceq lcc(c^*) \preceq lcc(f_2^*) \preceq lsc(e^*) \preceq ac(e)$. If $e \in hs(\Gamma^*, s)$ then there exists a simple cycle $\overline{c}$ formed by p and s, or by a part of them, such that e is enclosed by or is part of $\overline{c}$. In both cases the c-planarity of Γ^* implies that $ac(\overline{c}) \preceq ac(e)$. Also, $ac(p) \preceq ac(\overline{c})$ ($ac(p)$ is an ancestor of $ac(s)$ and of $ac(\overline{c})$) and since p is part of c $ac(c) \preceq ac(p)$. So $ac(c) \preceq ac(p) \preceq ac(\overline{c}) \preceq ac(e)$. □

Lemma 4 *Let μ be a P node such that the pertinent graphs of its children are c-planar. Suppose that: (i) it does not exist a set of three edges of $skeleton(\mu)$ that are pairwise incompatible, (ii) there exists at most one edge e^* of $skeleton(\mu)$ such that $lsc(e^*) \prec lcc(e^*)$, and (iii) if there exists e^* then each edge $e \neq e^*$ of $skeleton(\mu)$ is such that $ac(e) \preceq lsc(e^*)$. Then $pertinent(\mu)$ is c-planar*

Proof. First, observe that all the edges of $skeleton(\mu)$ are pairwise comparable since they share the poles. Subdivide the edges of $skeleton(\mu)$ into two ordered sets $I_L = \{l_1, l_2, \ldots, l_p\}$ and $I_R = \{r_1, r_2, \ldots, r_q\}$ such that all the edges in I_L (in I_R) are pairwise compatible. This partition exists since $skeleton(\mu)$ does not contain three pairwise incompatible edges. Also the edges in I_L (in I_R) are ordered so that $lcc(l_p) \preceq lcc(l_{p-1}) \preceq \ldots \preceq lcc(l_1)$ ($lcc(r_q) \preceq lcc(r_{q-1}) \preceq \ldots \preceq lcc(r_1)$). If two or more edges $l_i, l_{i+1}, \ldots, l_m$ ($r_j, r_{j+1}, \ldots, r_n$) in I_L (in I_R) have the same lowest connecting cluster, they are ordered so that $hsc(l_m) \preceq hsc(l_{m-1}) \preceq \ldots \preceq hsc(l_i)$ ($hsc(r_n) \preceq hsc(r_{n-1}) \preceq \ldots \preceq hsc(r_j)$). If two or more edges l_i (r_j) in I_L (in I_R) have the same lowest connecting cluster and the same highest side cluster, they are ordered in any way.

Consider the embedding Γ_μ of $skeleton(\mu)$ obtained by placing its edges in the order $I = \{l_p, \ldots, l_1, r_1, \ldots, r_q\}$. We show that $skeleton(\mu)$ is c-planar. This is done by considering three edges e_1, e_2, e_3 that appear in this order in Γ_μ and by proving that the lowest connecting cluster of the cycle $c = \{e_1, e_3\}$ is an ancestor of the lowest connecting cluster of e_2. If e_1 and e_3 are both in I_L (in I_R), then $lcc(c) = lcc(e_1) \preceq lcc(e_2)$ (resp. $lcc(c) = lcc(e_3) \preceq lcc(e_2)$). Otherwise $e_1 \in I_L$ and $e_3 \in I_R$. Suppose $e_2 \in I_L$ ($e_2 \in I_R$), $lcc(c) \preceq lcc(e_1) \preceq lcc(e_2)$ (resp. $lcc(c) \preceq lcc(e_3) \preceq lcc(e_2)$).

We now show that Γ_μ is such that each edge e of $skeleton(\mu)$ is incident to two faces f_1 and f_2 such that $lcc(f_1) \preceq hsc(e)$ and $lcc(f_2) \preceq lsc(e)$. The internal faces of Γ_μ consist of two edges that are consecutive in I. We denote by $\{e_1, e_2\}$ an internal face of Γ_μ between edges e_1 and e_2. We have that $lcc(\{l_i, l_{i+1}\}) = lcc(l_{i+1})$, for $1 \leq i < p$, and that $lcc(\{r_j, r_{j+1}\}) = lcc(r_{j+1})$, for $1 \leq j < q$. Since l_i and l_{i+1}, for $1 \leq i < p$ (r_j and r_{j+1}, for $1 \leq j < q$), are not incompatible, then $lcc(\{l_i, l_{i+1}\}) \preceq hsc(l_i)$ ($lcc(\{r_j, r_{j+1}\}) \preceq hsc(r_j)$). Also denoting by f_e the external face of Γ_μ we clearly have $lcc(f_e) \preceq hsc(l_p)$ and $lcc(f_e) \preceq hsc(r_q)$. By the second condition on the P nodes there exists at most one edge e^* such that $lsc(e^*) \prec lcc(e^*)$, and so $lcc(f_2) \preceq lsc(e)$ can be violated only for $e = e^*$. By the ordering of the edges in $\Gamma(\mu)$, we have that either $e^* = r_1$ or $e^* = l_1$. W.l.o.g. suppose that $e^* = r_1$. By the second condition on the P nodes $lcc(\{l_1, r_1\}) \preceq lcc(l_1) \preceq lsc(r_1)$. By Lemma 3 we can conclude that *pertinent*(μ) is c-planar. $\square$

Lemma 5 *Let $C(G, T)$ be a c-connected clustered graph and let (u, v) be an edge of G such that $ac(u, v)$ is the root of T. If $C(G - (u, v), T)$ admits a c-planar embedding with u and v on the external face, then C is c-planar.*

Proof. Since $ac((u, v))$ is the root of T, it is easy to see that (u, v) cannot create cycles enclosing an edge whose allocation cluster is a proper ancestor of the allocation cluster of the cycle. Hence, by Theorem 1, C admits a c-planar embedding obtained by adding (u, v) to the one of $C(G - (u, v), T)$. $\square$

The above lemma completes the proof of sufficiency.

5 Proof of the Necessity

The proof of the necessity of Theorem 2 is split into five lemmas, that, considered altogether, constitute a complete proof. Before giving such lemmas, we discuss the following issue: how limiting is the choice of rooting $\mathcal{T}$ to a specific edge whose allocation cluster is the root of T? The answer is in the following theorem.

Theorem 3 *Let $C(G, T)$ be a c-connected c-planar clustered graph and let Γ be any c-planar embedding of C. Let e be an edge whose allocation cluster is the root of T. Change the external face of Γ choosing as external any face containing e. The resulting planar embedding is still c-planar.*

Proof sketch. Let f (g) be the old (new) external face. Suppose that after the change of the external face there is a cycle c that encloses an edge e such that $ac(e) \prec ac(c)$. Such a cycle is external to f in the new embedding, otherwise even Γ would have been non c-planar. Hence, in the new embedding we have that g encloses c that encloses f. Recall that $ac(f) = ac(g)$ equal to the root of T. This implies that in the old embedding we had a cycle c enclosing an edge (of g) with higher allocation cluster. A contradiction. $\square$

Now we start the proof of the necessity. The first two lemmas are aimed to prove the necessity of the conditions on the skeletons of the R nodes.

Lemma 6 *Let μ be an R node of $\mathcal{T}$ such that the conditions of Theorem 2 are satisfied for all nodes of the subtree rooted at μ but for μ itself. Namely, suppose that the embedding of skeleton(μ) with its poles on the external face is not c-planar. Then C is not c-planar.*

Proof. Because of the non c-planarity, the unique embedding of $skeleton(\mu)$ with its poles u and v on the external face contains a cycle c^* that encloses an edge e^* such that $lcc(e^*) \prec lcc(c^*)$. It is possible to find in each embedding of $pertinent(\mu)$ a cycle c corresponding to c^* by substituting each virtual edge e^* of c^* with a lowest connecting path of e^*. By the definition of lowest connecting path, $ac(c) = lcc(c^*)$. Also, by the definition of $lcc(e^*)$ we have that $pertinent(e^*)$ contains an edge e with $ac(e) = lcc(e^*)$. Since μ is an R node, in any embedding of $pertinent(\mu)$ e lies inside c. So, because of Theorem 1, $pertinent(\mu)$ is not c-planar with (u, v) on the external face and C is not c-planar with the root of $\mathcal{T}$ on the external face. By Theorem 3 it follows that C is not c-planar. □

Lemma 7 *Let μ be an R node of $\mathcal{T}$ such that the conditions of Theorem 2 are satisfied for all nodes of the subtree rooted at μ but for μ itself. Namely, suppose that $skeleton(\mu)$ contains an edge e incident to two faces f_1 and f_2 ($lcc(f_1) \preceq lcc(f_2)$) with $hsc(e) \prec lcc(f_1)$ or $lsc(e) \prec lcc(f_2)$, then C is not c-planar.*

Proof. Concatenating the lowest connecting paths of the edges composing f_1 (f_2), we can find in $pertinent(\mu)$ a cycle c_1 (c_2) corresponding to f_1 (f_2) of $skeleton(\mu)$ with $ac(c_1) = lcc(f_1)$ ($ac(c_2) = lcc(f_2)$). Due to Property 7 there exist two edges e_1 and e_2 of $pertinent(e)$ such that $ac(e_1) = hsc(e)$, $ac(e_2) \preceq lsc(e)$ and that either e_1 lies inside c_1 and e_2 lies inside c_2 or e_1 lies inside c_2 and e_2 lies inside c_1. In both cases we have a cycle c that contains an edge e such that $ac(e) \prec ac(c)$, so $pertinent(\mu)$ is not c-planar with (u, v) on the external face and C is not c-planar with the root of $\mathcal{T}$ on the external face. By Theorem 3 it follows that C is not c-planar. □

The next three lemmas are to prove the necessity of the conditions of Theorem 2 on the skeletons of the P nodes.

Lemma 8 *Let μ be a P node of $\mathcal{T}$ such that the conditions of Theorem 2 are satisfied for all nodes of the subtree rooted at μ but for μ itself. Namely, suppose that $skeleton(\mu)$ contains three edges e_1, e_2, e_3 that are pairwise incompatible, then C is not c-planar.*

Proof. Consider any embedding of $skeleton(\mu)$ and suppose, w.l.o.g., that e_1, e_2, and e_3 are embedded in this order around the poles. Consider a cycle c of $pertinent(\mu)$ composed by a lowest connecting path of e_1 and by a lowest connecting path of e_3. We have that $lcc(c)$ is the lowest common ancestor of $lcc(e_1)$ and $lcc(e_3)$. By applying Property 7 it is possible to find an edge $e \in pertinent(e_2)$ such that $ac(e) = hsc(e_2)$. Since e_2 is incompatible with both e_1 and e_3 we have $ac(e) = hsc(e_2) \prec lcc(c)$. Hence, $pertinent(\mu)$ is not c-planar with (u, v) on the external face and C is not c-planar with the root of $\mathcal{T}$ on the external face. By Theorem 3 it follows that C is not c-planar. □

Lemma 9 *Let μ be a P node of $\mathcal{T}$ such that the conditions of Theorem 2 are satisfied for all nodes of the subtree rooted at μ but for μ itself. Namely, suppose that $skeleton(\mu)$ contains two edges e_1^* and e_2^* with $lsc(e_1^*) \prec lcc(e_1^*)$ and $lsc(e_2^*) \prec lcc(e_2^*)$, then C is not c-planar.*

Proof. Consider a lowest connecting path p_1 of e_1^* and a lowest connecting path p_2 of e_2^*. Let c be the cycle obtained by concatenating p_1 and p_2; we have that $lcc(c)$ is the lowest common ancestor of $lcc(e_1^*)$ and $lcc(e_2^*)$. W.l.o.g. we assume that $lcc(c) = lcc(e_1^*)$. By Property 7 in any embedding of $pertinent(e_1^*)$ there exist edges e_1 and e_2 that are separated by p_1 and are such that $ac(e_1) = hsc(e_1^*)$ and $ac(e_2) \preceq lsc(e_1^*)$. One between e_1 and e_2 is enclosed by c. By the above inequalities $ac(e_1) \prec lcc(c)$, $ac(e_2) \prec lcc(c)$, and hence $pertinent(\mu)$ is not c-planar with (u, v) on the external face and C is not c-planar with the root of $\mathcal{T}$ on the external face. By Theorem 3 it follows that C is not c-planar. □

Lemma 10 *Let μ be a P node of $\mathcal{T}$ such that the conditions of Theorem 2 are satisfied for all nodes of the subtree rooted at μ but for μ itself. Namely, suppose that $skeleton(\mu)$ contains an edge e^* with $lsc(e^*) \prec lcc(e^*)$ and an edge $e \neq e^*$ such that $lsc(e^*) \prec lcc(e)$, then C is not c-planar.*

Proof. By Property 5 we have that $hsc(e^*) \prec lcc(e)$. Consider a lowest connecting path p_{e^*} of e^*, a lowest connecting path p_e of e, and the cycle $c = p_{e^*} \cup p_e$; $lcc(c)$ is the lowest common ancestor of $lcc(e^*)$ and $lcc(e)$. By Property 7 in any embedding of $pertinent(e^*)$ there exist edges e_1 and e_2 that are separated by p_{e^*} and are such that $ac(e_1) = hsc(e^*)$ and $ac(e_2) \preceq lsc(e^*)$. One between e_1 and e_2 is enclosed by c. By the above inequalities $ac(e_1) \prec lcc(c)$, $ac(e_2) \prec lcc(c)$, and hence with (u, v) on the external face and C is not c-planar with the root of $\mathcal{T}$ on the external face. By Theorem 3 it follows that C is not c-planar. □

The above lemma completes the proof of necessity of Theorem 2.

6 Characterization of the C-Planarity of General C-connected Clustered Graphs

In this section we extend the characterization given in the previous sections to general c-connected clustered graphs.

Theorem 4 *Let $C(G, T)$ be a c-connected clustered graph and let $\mathcal{B}$ be the BC-tree of G rooted at a block ν that contains an edge e whose allocation cluster is the root of T. C is c-planar if and only if each block μ of $\mathcal{B}$ admits a c-planar embedding Γ_μ such that the parent cut-vertex of μ (if any) is on the external face of Γ_μ and each child cut-vertex ρ_i of μ is incident to a face f_i whose lowest connecting cluster is an ancestor of the allocation cluster of $pertinent(\rho_i)$.*

Proof. First, we show the sufficiency of the conditions. Suppose each block μ admits an embedding Γ_μ respecting the above conditions. We show how to build a c-planar embedding Γ_G of G, by suitably merging the embeddings Γ_μ. We traverse top-down $\mathcal{B}$ starting at its root ν. Let μ be the current block, and let $\mu_1, \mu_2, \dots, \mu_k$ be the blocks whose parent cutvertices $\rho_1, \rho_2, \dots, \rho_k$ (not necessarily distinct) are children of μ. We select a face f_i of Γ_μ incident to ρ_i and whose lowest connecting cluster is an ancestor of the allocation cluster of pertinent(ρ_i). We embed Γ_{μ_i} into f_i identifying the two instances of ρ_i in μ and μ_i. Distinct children of the same cutvertex are embedded in such a way that one does not enclose the other. Now we show that the obtained embedding Γ_G is c-planar. Suppose, by contradiction, that Γ_G is not c-planar. Let p be a simple cycle enclosing an edge e with $ac(e) \prec ac(p)$. Observe that, since p is simple, all edges of p are contained into the same block μ^*. If e also belongs to μ^*, then Γ_{μ^*} is not c-planar, contradicting the hypothesis. Otherwise, suppose e belongs to pertinent(ρ_i), where ρ_i is a child cutvertex of μ^*. Hence, $ac(pertinent(\rho_i)) \preceq ac(e)$. By construction, $ac(f_i) \preceq ac(pertinent(\rho_i))$. Since each edge of f_i belongs or is internal to p, Theorem 1 ensures that $ac(p) \preceq ac(f_i)$. Therefore, we have $ac(p) \preceq ac(e)$ contradicting the hypothesis that $ac(e) \prec ac(p)$.

It is trivial that the c-planarity of the blocks of the BC-tree is a necessary condition for the c-planarity of C. Also by Theorem 3 the choice of rooting $\mathcal{B}$ to a node ν containing an edge e whose allocation cluster is the root of T, and the choice of rooting $\mathcal{T}_\nu$ to e are not limiting, that is if a c-planar embedding Γ_G of G exists, then there exists also a c-planar embedding of G with e on the external face. This leads to assume that Γ_G has e on the external face. In order to show the necessity of the other two conditions, suppose that a block $\mu^* \neq \nu$ does not admit a c-planar embedding with its parent cutvertex on the external face. Then the blocks that are ancestor of μ^* have to be embedded inside μ^* and so e cannot be on the external face of Γ_G, contradicting the hypothesis. Now suppose that a child cutvertex ρ_i of μ is such that $ac(pertinent(\rho_i)) \prec ac(f_j)$ for each face f_j of Γ_μ incident to ρ_i. By Property 2 $pertinent(\rho_i)$ contains an edge e_i such that $ac(e_i) = ac(pertinent(\rho_i))$. So embedding $pertinent(\rho_i)$ inside μ creates a cycle f_j containing an edge e_i such that $ac(e_i) \prec ac(f_j)$, while embedding μ inside any embedding of $pertinent(\rho_i)$ implies that e cannot be on the external face of Γ_G, contradicting the hypothesis. Hence, C is not c-planar with e on the external face and by Theorem 3 it follows that C is not c-planar. □

References

[1] S. Cornelsen and D. Wagner. Completely connected clustered graphs. In *Proc. WG*, volume 2880 of *LNCS*. Springer-Verlag, 2003.

[2] P. F. Cortese, G. Di Battista, F. Frati, M. Patrignani, and M. Pizzonia. C-planarity of c-connected clustered graphs: Part II – Testing and embedding algorithm. Tech. Report RT-DIA-110-2006, Dip. Informatica e Automazione, Univ. Roma Tre, Jun 2006. `http://web.dia.uniroma3.it/ricerca/rapporti/rapporti.php`.

[3] E. Dahlhaus. Linear time algorithm to recognize clustered planar graphs and its parallelization. In C.L. Lucchesi, editor, *LATIN ́98*, volume 1380 of *LNCS*, 1998.

[4] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing*. Prentice Hall, Upper Saddle River, NJ, 1999.

[5] G. Di Battista and R. Tamassia. On-line planarity testing. *SIAM J. Comput.*, 25:956–997, 1996.

[6] S. Even. *Graph Algorithms*. Computer Science Press, Potomac, Maryland, 1979.

[7] Q. W. Feng, R. F. Cohen, and P. Eades. Planarity for clustered graphs. In *Proc. ESA '95*, volume 979 of *LNCS*, pages 213–226. Springer-Verlag, 1995.

[8] T. Lengauer. Hierarchical planarity testing algorithms. *J. ACM*, 36(3), 1989.