Università degli Studi di Roma Tre
Dipartimento di Informatica e Automazione
Via della Vasca Navale, 79 – 00146 Roma, Italy

On Embedding a Cycle in a Plane Graph

P. F. Cortese, G. Di Battista, M. Patrignani, and M. Pizzonia

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Dipartimento di Informatica e Automazione,
Università di Roma Tre,
Rome, Italy.
{cortese,gdb,patrigna,pizzonia}@dia.uniroma3.it

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ABSTRACT

Consider a planar drawing Γ of a planar graph G such that the vertices are drawn as small circles and the edges are drawn as thin strips. Consider a cycle c of G. Is it possible to draw c as a non-intersecting closed curve inside Γ, following the circles that correspond in Γ to the vertices of c and the strips that connect them? We show that this test can be done in polynomial time and study this problem in the framework of clustered planarity for highly non-connected clustered graphs.

1 Introduction

Let Γ be a planar drawing of a planar graph G and c be a cycle composed by vertices and edges of G. We deal with the problem of testing if c can be drawn on Γ without crossings.

Of course, if the vertices of G are drawn as points, the edges as simple curves, and the drawing of c must coincide with the drawing of its vertices and edges, then the problem is trivial. In this case c can be drawn without crossings if and only if it is simple.

We consider the problem from a different point of view. Namely, we suppose that the vertices of G are drawn in Γ as "small circles" and the edges as "thin strips". Hence, c can pass several times through a vertex or through an edge without crossing itself. In this case even a non-simple cycle can have a chance to be drawn without crossings. For example, the cycle of Fig. 1.a can be drawn without crossings, while the cycle of Fig. 1.b cannot.

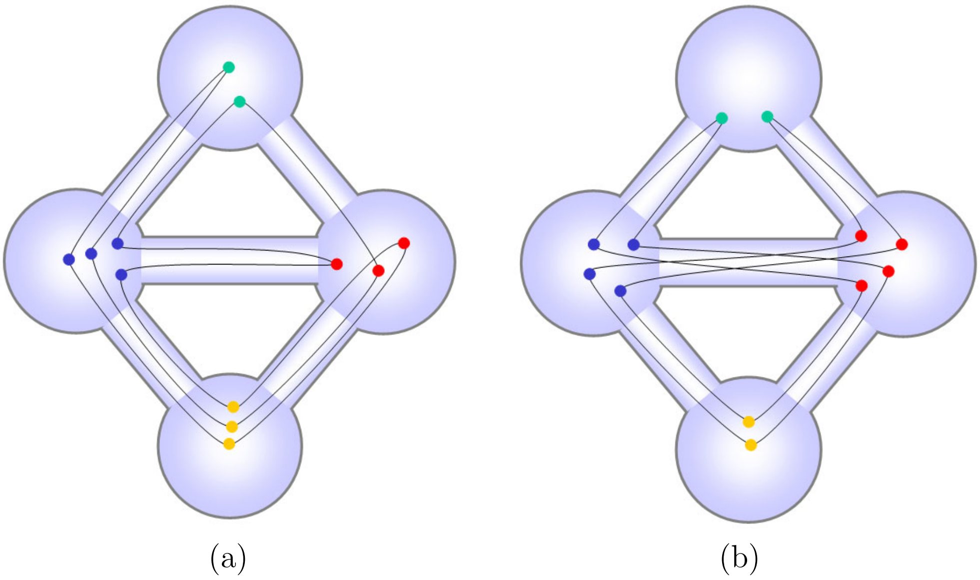


Figure 1: A cycle which could be drawn without crossings (a) and a cycle which cannot (b).

The problem, in our opinion, is interesting in itself. However, we study it because of its meaning in the field of *clustered planarity* [11, 10].

Clustered planarity is a classical Graph Drawing topic (see [4] for a survey). A *cluster* of a graph is a non empty subset of its vertices. A *clustered graph* $C(G, T)$ is a graph G plus a rooted tree T such that the leaves of T are the vertices of G. Each node ν of T corresponds to the cluster $V(\nu)$ of G whose vertices are the leaves of the subtree rooted at ν. The subgraph of G induced by $V(\nu)$ is denoted as $G(\nu)$. An edge e between a vertex of $V(\nu)$ and a vertex of $V - V(\nu)$ is *incident* on ν. Graph G and tree T are called *underlying graph* and *inclusion tree*, respectively. A clustered graph is *connected* if for each node ν of T we have that $G(\nu)$ is connected.

In a *drawing* of a clustered graph vertices and edges of G are drawn as points and curves as usual [8], and each node ν of T is a simple closed region $R(\nu)$ such that: (i) $R(\nu)$ contains the drawing of $G(\nu)$; and (ii) $R(\nu)$ contains a region $R(\mu)$ if and only if μ is a descendant of ν in T. We say that edge e and region R have an *edge-region crossing* if both endpoints of e are outside R and e crosses the boundary of R. A drawing of a clustered graph is *c-planar* if it does not have edge crossings and edge-region crossings. A clustered graph is *c-planar* if it has a c-planar drawing. C-planarity testing algorithms for connected clustered graphs are shown in [13, 11, 6]. A planarization algorithm for connected clustered graph is shown in [7].

However, the complexity of the c-planarity testing for a non connected clustered graph is still unknown. A contribution on this topic has been given by Gutwenger et al. that presented a polynomial time algorithm for c-planarity testing for *almost connected* clustered graphs [12].

Another contribution studying the interplay between c-planarity and connectivity has been presented in [3] by Cornelsen and Wagner. They show that a *completely connected* clustered graph is c-planar if and only if its underlying graph is planar. A completely connected clustered graph is so that not only each cluster is connected but also its complement is connected.

A clustered graph $C(G, T)$ is *flat* if all the leaves of T have distance two from the root. This implies that all the non-root clusters have depth 1 in T. Hence, in a flat clustered graph $C(G, T)$ a *graph of the clusters* $G^1(C)$ can be identified. Vertices of $G^1(C)$ are the children of the root of T and an edge (μ, ν) exists if and only if an edge of G exists incident to both μ and ν.

Flat clustered graphs offer a way to deepen our insight into the properties of non-connected c-planar clustered graphs. In fact, by changing the families of the graphs G and $G^1(C)$, c-planarity problems of increasing complexity can be identified. The works in [2, 1] by Biedl, Kaufmann, and Mutzel can be interpreted as a linear time c-planarity test for non connected flat clustered graphs with exactly two clusters.

A *clustered cycle* is a flat clustered graph whose underlying graph is a cycle. In [5] it is shown that for a clustered cycle $C(G, T)$ where $G^1(C)$ is also a cycle, the c-planarity testing and embedding problem can be solved in linear time.

A *rigid clustered cycle* is a clustered cycle C in which $G^1(C)$ has a prescribed planar embedding. In this paper we tackle the c-planarity testing and embedding problem for rigid clustered cycles. Namely, consider again the problem stated at the beginning of this section according to the above definitions. The cycle is the underlying graph of a flat clustered graph and the nodes of the graph are the clusters. If you are able to find a drawing of the cycle without intersections you are also able to find a c-planar embedding for the rigid clustered cycle and vice versa.

In this paper we present the following results.

- We develop a new theory for dealing with rigid clustered cycles, based on operations that preserve their c-planarity (Section 3).
- We show that the c-planarity of a rigid clustered cycle can be tested in polynomial time (Section 4). As a side effect we also solve in polynomial time the cycle drawing problem stated at the beginning of the section.
- If the rigid clustered cycle is c-planar we also show a simple method for computing a planar embedding of it (Section 5).

Section 2 contains basic definitions, while conclusions and open problems are in Section 6.

2 Basic Definitions

We assume familiarity with connectivity and planarity of graphs [9, 8].

In the following we need a slightly wider definition of flat clustered graph with respect to the one generally used in literature and recalled in Section 1. In particular, we allow $G^1(C)$ to have multiple edges between two nodes. Hence, in a *flat clustered graph* $C(G, G^1)$, G^1 is a graph, possibly with multiple edges, and G is a graph where each vertex belongs to a cluster and each edge belongs to an edge of G^1. In the following, to avoid ambiguities, edges of $G^1(C)$ will be called *pipes* while vertices will be called *nodes* or *clusters*.

Given a cluster $\mu \in G^1(C)$, we denote by $deg(\mu)$ the number of pipes that are adjacent to μ in $G^1(C)$, where multiple pipes count for their multiplicity. The *size* of a pipe of $G^1(C)$ is the number of edges of G it contains.

It is easy to see that a path in G whose vertices belong to the same cluster can be collapsed into a single vertex without affecting the c-planarity property of the clustered cycle. Hence, in the following we consider only clustered cycles where consecutive vertices belong to distinct clusters.

Consider a vertex v of G and suppose it is incident to edges (u, v) and (v, w). Suppose that both (u, v) and (v, w) belong to the same pipe of $G^1(C)$. We say that v is a *cusp*.

Given a rigid clustered cycle C the *embedding* of C is the specification, for each pipe a in $G^1(C)$ and for each end node μ of a, of the total ordering of the edges contained in a when turning around μ clockwise. An embedding of a clustered cycle is c-planar if there exists a planar drawing of C that respects such embedding. To describe a c-planar embedding is sufficient to specify for each pipe the order of its edges with respect to one of its end nodes only.

3 Fountain Clusters

Consider a clustered cycle C and one of its clusters $\mu = \{v_1, \dots, v_q\}$. For each v_i let w_i and z_i be its neighbors. Cluster μ is a *fountain cluster* if there exists a cluster ν different from μ such that for each v_i we have that $w_i \in \nu$ or $z_i \in \nu$. (see Fig. 2 for an example). We call *base* of μ the pipe of $G^1(C)$ between μ and ν.

Property 1 *If $deg(\mu) = 2$, then μ has two bases. In all other cases μ has one base.*

A *fountain clustered cycle* is a clustered cycle in which each cluster is a fountain cluster.

Property 2 *Let μ be a fountain cluster, and v be a cusp of μ. The edges incident to v belong to the base of μ.*

3.1 Cluster Expansion

Given a cluster μ of C, we call *cluster expansion* of μ the following operation (see Fig. 3), that produces the clustered cycle C'.

Let $a_1, \dots, a_k$ be the pipes incident to μ, where $k = deg(\mu)$. Let v a vertex belonging to μ, and let e_i and e_j be the edges incident to v, where $e_i \in a_i$ and $e_j \in a_j$, respectively. Note that if v is a cusp, then $i = j$.

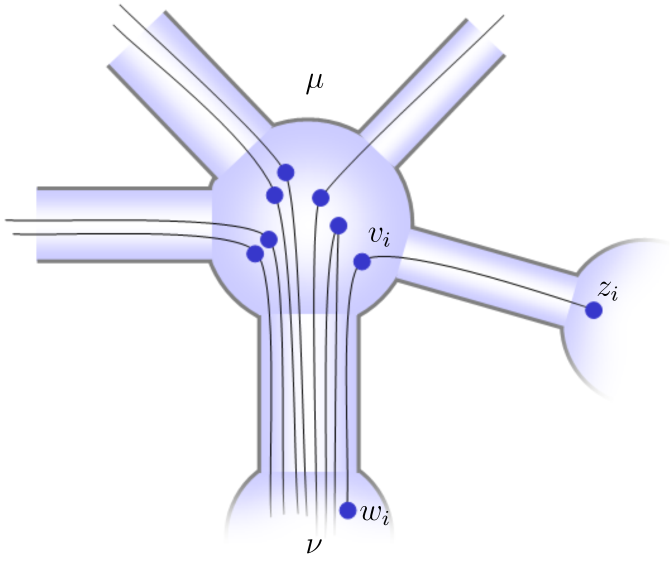


Figure 2: A fountain cluster.

Cluster μ is replaced in C' with k new clusters $\mu_1, \dots \mu_k$, each one incident to pipes $a_1, \dots, a_k$, respectively. All the other clusters of C are unchanged in C'.

Each non-cusp vertex v in μ having edges e_i and e_j (note that in this case $i \neq j$) is represented in C' by two new vertices v' and v''.

Vertex v' belongs to μ_i and vertex v'' belongs to μ_j, e_i is incident to v', e_j is incident to v'' and a new edge is added between v' and v''.

Each cusp vertex v having its edges in pipe a_i stays unchanged in C', and belongs to cluster μ_i.

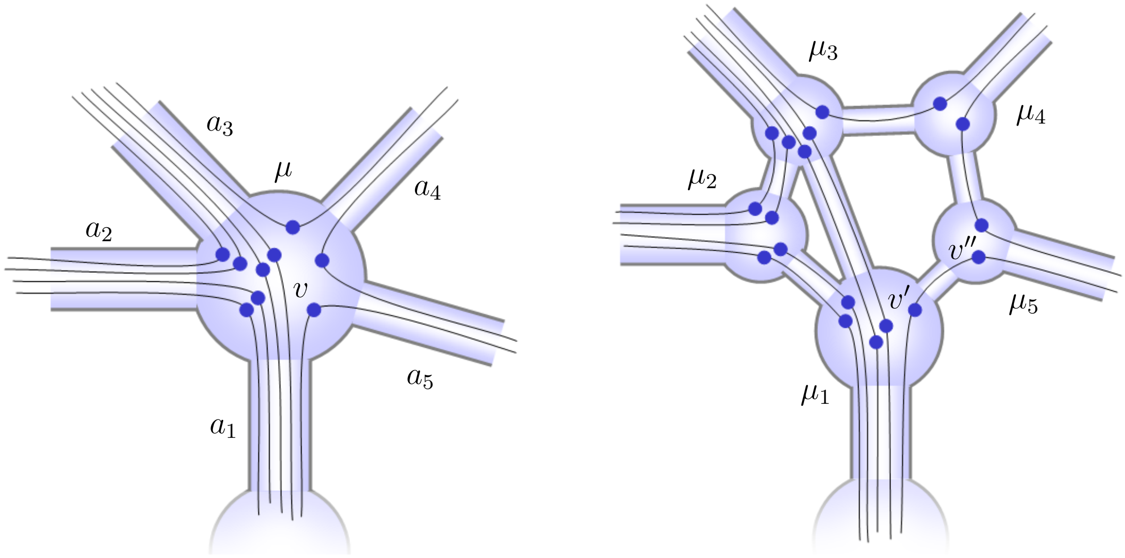
Note that the clusters produced by a cluster expansion are all fountain clusters. Hence, after one expansion the number of non-fountain clusters of C' is not greater than the number of non-fountain clusters of C. Also, before applying the cluster expansion, μ could be the end node of multiple pipes. After the cluster expansion such multiple pipes are eliminated.

Up to now the expansion operation has been defined whithout considering the embedding of C and C'. If C is embedded (rigid) it is easy to extend the definition considering also embedding issues. Namely, we embed the new pipes around the new nodes with the same order the old edges had in C. Note that, even if the starting embedding is planar, the resulting embedding may be not planar due to th new pipes inserted among the clusters $\mu_1, \dots \mu_k$.

Given a rigid clustered cycle C, a cluster expansion of one of its clusters μ is *feasible* if the embedding induced on $G^1(C')$ is planar, that is, if C' is a rigid clustered cycle.

Lemma 1 *Given a rigid clustered cycle C, if a cluster expansion of one of its clusters μ is not feasible, then C is not c-planar.*

Proof: If the cluster expansion of μ is not feasible, then the induced embedding on $G^1(C')$ contains a crossing, that is, it contains two pipes (μ_i, μ_k) and (μ_j, μ_l), with $i < j < k < l$. This implies that there exist at least two paths of G, one traversing clusters ν_i, μ, ν_k and the other traversing ν_j, μ, ν_l. Since the embedding of μ is fixed, this two paths cannot be drawn without intersections. $\square$



(a) (b)

Figure 3: An example of cluster expansion: (a) A non-fountain cluster μ. (b) The result of the cluster expansion.

A *cluster expansion* operation on a clustered cycle C is done performing a cluster expansion for each non-fountain cluster of C. A cluster expansion of a rigid clustered cycle is *feasible* if all the required cluster expansions are feasible, that is if the result is a rigid clustered cycle.

Property 3 *The cluster expansion of a clustered cycle produces a fountain clustered cycle.*

Lemma 2 *Let C be a rigid clustered cycle and let μ be a cluster of C. Let C' be the result of a feasible cluster expansion applied to μ. C is c-planar iff C' is c-planar.*

Proof: (sketch) Suppose that C is c-planar, and let Γ be a c-planar embedding of C. A c-planar embedding Γ' of C' can be computed as follows. For each pipe that is present both in C and C', including pipes $a_1, \ldots, a_k$, we assume that the order of edges in Γ' is the same as in Γ. All other pipes, added among nodes $\mu_1, \ldots, \mu_k$ by the expansion, contain edges which are adjacent to edges in $a_1, \ldots, a_k$, whose embedding has been estabilished. Hence, the order of the edges in such pipes is induced by the order of the edges in pipes $a_1, \ldots, a_k$, and the c-planarity of Γ' follows from the c-planarity of Γ.

Suppose now that C' is c-planar, and let Γ' be a c-planar embedding of C'. A c-planar embedding Γ of C can directly obtained from Γ'. Since all pipes of C are also present in C', the order of their edges can be assumed to be the same as in Γ'. Consider edge e of pipe (μ_i, μ_j) in Γ'. The path e_i, e, e_j of Γ', where $e_i \in a_i$ and $e_j \in a_j$ corresponds to path e_i, e_j in Γ. Hence, the c-planarity of Γ' implies the c-planarity of Γ. $\square$

Lemma 3 *Let C be a rigid clustered cycle and let C' be a feasible cluster expansion of C. C is c-planar iff C' is c-planar.*

Proof: (sketch) A feasible cluster expansion of C is obtained by repeatedly applying feasible cluster expansions to its non-fountain clusters. From Lemma 2 all such expansions preserve c-planarity. $\square$

3.2 Pipe Contraction

We define the *pipe contraction* operation on a clustered cycle C with no multiple pipes in $G^1(C)$ as follows. Let b be a pipe incident to fountain clusters μ and ν, such that b is base for μ and for ν.

The pipe contraction removes μ, ν, and b, and adds a new cluster μ', which is adjacent to all the clusters which μ and ν were adjacent to. If μ and ν were adjacent to the same cluster ρ, μ' is doubly adjacent to ρ; that is the resulting clustered cycle may have multiple pipes in $G^1(C)$. All the edges of G corresponding to pipe b are contracted, and each edge which is not incident to a cusp produces a vertex in μ' as well as each sequence of edges separated by cusps.

An example of pipe contraction is shown in Fig. 4. Note that the new cluster μ' is, in general, not a fountain cluster. If C has a prescribed embedding we assume that the result has also a prescribed embedding in which the circular order of the pipes around μ' is the same as the circular order they have in C around the subgraph constituted by μ, ν, and b.

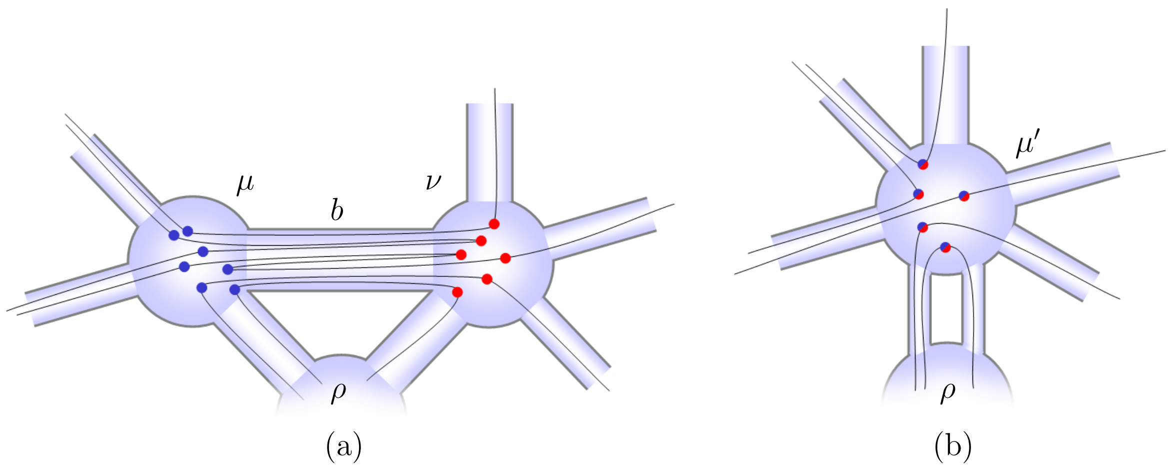


Figure 4: An example of pipe contraction: (a) pipe b before contraction; (b) The result of the contraction of b.

Lemma 4 *Let C be a fountain clustered cycle and C' be obtained from C by applying a pipe contraction operation. C is c-planar iff C' is c-planar.*

Proof: (sketch) Suppose that C is c-planar, let Γ be a c-planar embedding of C, we provide a c-planar embedding Γ' of C'. Let b be the contracted pipe and let μ and ν be the clusters which b is incident to. Moreover, let μ' be the new cluster in C' that replaces μ, ν, and b. All the pipes in C' are also present in C, and we assume for them the same order of the edges they contain. To prove that Γ' is planar note that each vertex v in μ' corresponds to two vertices in C, one in μ and one in ν, connected by a path p which may be a single edge or a sequence of cusps (see Fig. 4). The edges incident to v in C' are the same of the edges that are incident to the extremes of p in C. The c-planarity of Γ' follows from the planarity of Γ.

Suppose now that C' is c-planar, and let Γ' be a c-planar embedding of C'. We provide a c-planar embedding Γ of C. For each pipe in C which has a corresponding pipe in C' we assume the same order of the edges it contains. The only pipe of C which is not contained in C' is b.

In order to determine the order of edges of b, consider first the case in which no cusp belongs to b. Each edge $e \in b$ is incident to a vertex v_μ of μ and to a vertex v_ν of ν. Since there are no cusps, vertex v_μ (v_ν) has a second incident edge e_μ (e_ν) belonging to a pipe of μ (ν) different from b. Hence, the order of the edges of b is induced by the embedding of their adjacent edges in the pipes incident to μ (ν) and distinct from b. Since Γ' is planar, the edges in the pipes incident to μ and the edges in the pipes incident to ν induce the same order for the edges in b. This proves the c-planarity of Γ.

Now consider the case in which some cusps belongs to b. Let $e_1, e_2, \ldots, e_h$, $h \geq 4$, be a sequence of edges such that $e_2, \ldots, e_{h-1}$ belong to b, while e_1 and e_h belong to some other pipe adjacent to μ or ν. The order of e_2 and e_{h-1} in b is induced by the embedding of e_1 and e_h. The edges between e_2 and e_{h-1}, if any, can be inserted in b, following e_2, in the same order as they appear in the sequence, without loosing c-planarity. □

4 C-Planarity Testing of Clustered Cycles

In this section we describe a c-planarity testing algorithm for rigid clustered cycles. The following lemmas state properties of clustered cycles which are needed to prove the correctness of the algorithm.

Lemma 5 *Let C be a fountain clustered cycle. One of the following is true.*

1. *$G^1(C)$ is a simple cycle.*
2. *$G^1(C)$ is a simple path.*
3. *There exists at least one pipe in $G^1(C)$ which is a base for both its incident nodes and which is incident to a node with degree greater than two.*

Proof: (sketch) Suppose that both case 1 and case 2 of the statement of the lemma are false, and suppose, for a contradiction, that it does not exist in $G^1(C)$ a pipe which is a base for both its adjacent nodes and such that at least one of them has degree greater than two.

Let μ be a (fountain) cluster of C with $deg(\mu) > 2$ and let b be its base. Consider the cluster μ_1 that is opposite to μ with respect to b. By the fact that b is not a base for ν_1 and by Property 1 it follows that $deg(\mu_1) > 2$.

Let b_1 the base of μ_1. Since $deg(\mu_1) > 2$ and all pipes distinct from b_1 do not contain cusps (Property 2), then $size(b_1) > size(b)$. Consider now cluster μ_2, that is opposite to μ_1 with respect to b_1. Since b_1 is not a base for μ_2, then $deg(\mu_2) > 2$ and μ_2 has another base b_2, and $size(b_2) > size(b_1)$, and so on. Hence, since the number of clusters is finite, there exists a cluster μ_j such that its base b_j is incident to a cluster μ_i, with $i < j$. Then we have that μ_i is incident to a pipe b_j with $size(b_j) > size(b_i)$. This is a contradiction, because in a fountain cluster the size of a base is greater than or equal to the size of all the other pipes. □

We introduce now a quantity that will be used to analyze the algorithm both in terms of correctness and in terms of time complexity. Intuitively, it is an indicator of the structural complexity of $G^1(C)$. We denote by $\mathcal{E}(C)$ the following quantity:

$$\mathcal{E}(C) = \sum_{a \in \{\text{pipes of } G^1(C)\}} (size(a))^2.$$

We now concentrate on a pair of consecutive contraction-expansion operations and show how $\mathcal{E}$ changes. Let C be a fountain clustered cycle. Let b a pipe that is at the same time base of two distinct clusters μ and ν. Suppose that $deg(\mu) > 2$. Apply a pipe contraction to b. Let μ' be the cluster produced by the contraction. Apply, if feasible, a cluster expansion of μ'. Let C'' the obtained clustered cycle. We have that:

Lemma 6 *$\mathcal{E}(C) > \mathcal{E}(C'')$.*

Proof: Let C' be the clustered cycle generated by the pipe contraction applied to b. C' contains all the pipes of C but b, then $\mathcal{E}(C') = \mathcal{E}(C) - (size(b))^2$.

The cluster expansion applied to μ' produces a new clustered cycle C'', which has the same pipes of C' plus a set of new pipes $a_1, \dots, a_k$. It follows that $\mathcal{E}(C'') = \mathcal{E}(C') + \sum_{j=1}^{k}(size(a_j))^2 = \mathcal{E}(C) - (size(b))^2 + \sum_{j=1}^{k}(size(a_k))^2$. Observe that each edge contained in the pipes $a_1, \dots, a_k$ is generated by the split of a vertex in μ', and that the number of vertices in μ' is at most $size(b)$. Then, $\sum_{j=1}^{k} size(a_j) \leq size(b)$. Since $deg(\mu) > 2$ we have that $k > 1$. Hence, $\sum_{j=1}^{k}(size(a_k))^2 < (size(b))^2$, and the statement follows. □

Lemma 7 *A clustered cycle C whose graph of the clusters $G^1(C)$ is a path is c-planar.*

Proof: (sketch) Let $\mu_1, \dots, \mu_m$ be the nodes of $G^1(C)$ in the order in which they appear in the path. A planar embedding of C can be built as follows. Traverse the cycle G starting from a vertex in μ_1. Each edge of the cycle encountered in such traversal is positioned into its pipe. If in such pipe there were already positioned edges, insert the edge after the last inserted one. In inserting the edges the insertions must be performed always on the same side of the path. It is easy to see that when the path comes back to μ_1 for the last time it can always be connected to the starting point. □

We are now ready to introduce the c-planarity testing algorithm for a rigid clustered cycle C. First, the algorithm performs a cluster expansion for each non fountain cluster. If one of such expansions is not feasible, then, according to Lemma 1, C is not c-planar. If all the expansions are feasible, according to Property 3, C is now a fountain clustered cycle. If the clusters of C form a cycle, then the c-planarity can be easily tested using the results described in [5]. If $G^1(C)$ is a path, then Lemma 7 states that C is always c-planar. If the clusters of C form neither a cycle nor a path, then Lemma 5 ensures that there exists a pipe b which is a base for both its incident nodes and which is incident to a node with degree greater than two. Perform a contraction operation on b. Perform a cluster expansion on the resulting cluster. These last two steps are performed until C is a cycle, or a path, or a cluster expansion fails. Note that a pipe contraction may temporarily generate multiple pipes in $G^1(C)$; however, the subsequent cluster expansion produces a new clustered cycle which has no multiple pipes. The algorithm, called *ClusteredCyclePlanarityTesting*, is formally described in Figure 5.

Theorem 1 *There exists a polynomial time algorithm to test if a rigid clustered cycle is c-planar.*

Algorithm *ClusteredCyclePlanarityTesting*

input A rigid clustered cycle C

output True if C is c-planar, false otherwise

for all clusters μ in C **do**
  if μ is not a fountain cluster **then**
    perform a cluster expansion of μ
    if the cluster expansion of μ is not feasible **then**
      return false;
    end if
  end if
end for
$\{$at this point C is a fountain clustered cycle$\}$
while C is not a cycle or a path **do**
  let b a pipe of C incident to a cluster ν such that b is base for both its end nodes and $deg(\nu) > 2$
  apply a pipe contraction to b, obtaining cluster ν'.
  perform a cluster expansion of ν'
  if the cluster expansion of ν' is not feasible **then**
    return false
  end if
end while
$\{$at this point C is a cycle or a path$\}$
if C is a cycle **then**
  return the result of the c-planarity testing on C
else
  return true
end if

Figure 5: The algorithm for testing the c-planarity of a rigid clustered cycle.

Proof: First, we prove that algorithm ClusteredCyclePlanarityTesting can be always executed in a polynomial number of steps. Let C be a rigid clustered cycle whose underlying cycle is G and be n the number of vertices of G. In the first phase of the algorithm a cluster expansion is performed for all the non-fountain clusters. Each cluster expansion can be performed in polynomial time. At the end of this phase the number of vertices is at most $2n$. Suppose that $\overline{E}$ is the value of $\mathcal{E}(C)$ at the end of this phase. We have that $\overline{E} = O(n^2)$.

By Lemma 6 each pair of pipe contraction and cluster expansion decreases $\mathcal{E}(C)$ of at least one unit. Hence, the body of the **while** cycle is executed at most $\overline{E}$ times. Also, the base pipe that is used to perform the contraction can be determined in constant time using a suitable data structure that contains the candidate bases and that is updated after each operation. This proves that algorithm ClusteredCyclePlanarityTesting terminates in polynomial time.

Second, we prove that algorithm ClusteredCyclePlanarityTesting gives the correct result. Lemmas 2, 3, and 4 guarantee that the cluster expansion and pipe contraction operations can be applied without modifying the c-planarity property of the graph, while if a cluster expansion is not feasible the graph is not c-planar. If none of the cluster expansions fails, either the algorithm produces a k-cluster cycle and applies the c-planarity testing algorithm shown in [5], or produces a clustered path, which by Lemma 7 is always c-planar. Also, (see the above discussion) the algorithm always terminates. □

5 Computing C-Planar Embeddings of Clustered Cycles

In this section we show how to build an embedding for a c-planar rigid clustered cycle. We assume that Algorithm ClusteredCyclePlanarityTesting, described in Section 4, has been applied, and that each step of the algorithm has been recorded. The clustered cycle C_{end} obtained at the last step of the execution of that algorithm is such that $G^1(C_{end})$ is a cycle or a path. A c-planar embedding of C_{end} can be easily computed by using the results described in [5], if $G^1(C_{end})$ is a cycle, or by using the technique introduced in the proof of Lemma 7, if $G^1(C_{end})$ is a path.

The embedding of the input clustered cycle can be obtained by going through the transformations operated by Algorithm ClusteredCyclePlanarityTesting in reverse order starting from a c-planar embedding of C_{end}. Algorithm ClusteredCyclePlanarityTesting performs two kind of operations: pipe contraction and cluster expansion.

For each cluster expansion on a clustered cycle C, which produces a cluster cycle C', the embedding of C is directly obtained from the embedding of C' as described in the proof of Lemma 2 since all pipes in C' are also in C and their embedding do not change.

For each pipe contraction on a clustered cycle C, which produces a cluster cycle C', only part of the embedding of C can be directly obtained from the embedding of C' since C has one more pipe (the contracted one) with respect to C'. The proof of Lemma 4 describes how to compute a c-planar embedding of C starting from a c-planar embedding of C'.

From the above discussion and from the fact that ClusteredCyclePlanarityTesting has a polynomial time complexity we can state the following result.

Theorem 2 *Given a c-planar rigid clustered cycle, a c-planar embedding of it can be computed in polynomial time.*

6 Conclusions

In this paper we addressed the problem of drawing, without crossings, a cycle in a planar embedded graph and have shown that the problem can be solved in polynomial time.

If we interpret the problem and the result from the clustered planarity perspective it turns out that we have identified a new family of flat clustered graphs that are highly non-connected and whose c-planarity can be tested in polynomial time. This might be useful for deepening the insight into the general problem of testing the c-planarity of non-connected clustered graphs, whose computational complexity is still unknown.

However, a trivial generalization of the result to flat clustered graphs whose underlying graph is a general graph fails. In fact, suppose that the underlying graph is not a cycle but a more complex graph containing several cycles. One could argue that testing the c-planarity of all the cycles of the underlying graph would be sufficient for proving the c-planarity of the whole graph. Unfortunately, the condition is necessary but not sufficient. A counterexample is shown in Fig. 6. The underlying graph of the figure has 3 cycles. Each of them is c-planar while the whole graph is not.

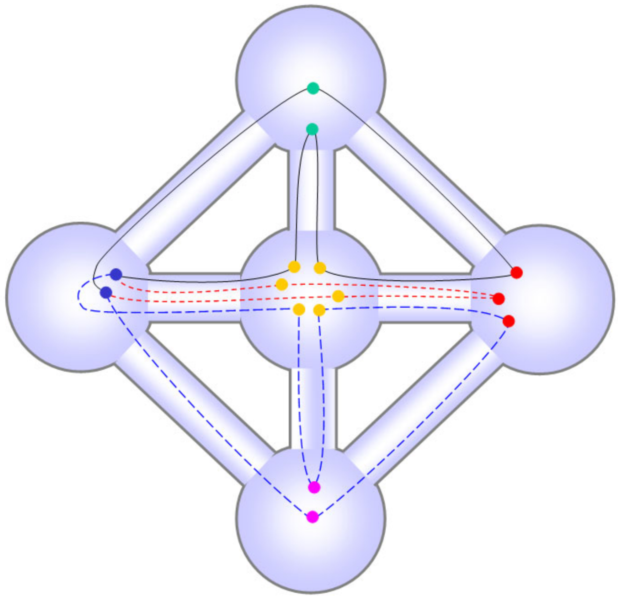


Figure 6: An example that shows that the c-planarity for cycles is only a necessary condition but not a sufficient one for the c-planarity of more complex graphs. The graph of the clusters is supposed to have fixed embedding while the underlying graph is planar and composed by three paths between two vertices (each path is drawn in the picture with a different line-style).

We believe there are other simplified versions of the c-planarity testing problem for flat clustered graphs that are worth investigating. For example, what if the underlying graph (or the graph of the clusters, or both) are equipped with a prescribed planar embedding? What if the underlying graph is a specific type of graph, e.g. it is a tree or a series-parallel graph?

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