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Simple Bounds for the Minimum Dominating Trail Set problem

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ABSTRACT

Given a graph G, the Minimum Dominating Trail Set ($MDTS$) problem consists in finding a minimum cardinality collection of pairwise edge-disjoint trails such that each edge of G has at least one endvertex on some trail. The $MDTS$ problem is NP–hard for general graphs. In this paper lower and upper bounds for the $MDTS$ problem on general graphs are presented.

1 Introduction

Given a graph $G = (V, E)$, a *trail* is a sequence $t = (v_0, e_0, v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k)$, where $(v_0, v_1, v_2, \ldots, v_k)$ are nodes of G, $(e_0, e_1, e_2, \ldots, e_{k-1})$ are distinct edges of G, and v_i and v_{i+1} are the endpoints of e_i for $0 \leq i \leq k-1$. In other words, a trail is a walk that can pass more times through the same node. A trail may consist of a single node.

A *dominating trail* in G is a trail such that each edge of G has at least one endpoint belonging to it (i.e., a dominating trail *covers* all the edges of G). Note that a dominating trail may not exist on G. A *dominating trail set* Σ is a collection of edge-disjoint trails that altogether cover all the edges of G. A *minimum dominating trail set* ($MDTS$) is a dominating trail set of minimum cardinality.

The $MDTS$ problem is related to the *Hamiltonian completion number* (HCN), which measures the non–hamiltonicity of a graph. A graph G is called Hamiltonian if it has a Hamiltonian path. The Hamiltonian completion number of a graph, usually denoted as $HCN(G)$, is the problem of finding the minimum number of edges which must be added to G to make it Hamiltonian. In particular, the $MDTS$ problem is related to $HCN(G)$ restricted to a particular class of graphs, called *line graphs*. The line graph $L(G)$ of $G = (V, E)$ is a graph having $|E|$ nodes, each node of $L(G)$ being associated to an edge of G. There is an edge between two nodes of $L(G)$ if the corresponding edges of G are adjacent. Harary and Nash-Williams [14] link the problem of finding $HCN(L(G))$ and the $MDTS$ of G showing that the line graph $L(G)$ has a Hamiltonian path if and only if G has a dominating trail. As a consequence, if $HCN(L(G)) = k$ then the cardinality of $MDTS$ of G is $k + 1$.

Another problem related to the $MDTS$ problem is that of finding a *dominating cycle* in a graph. A dominating cycle C in a graph G is a cycle such that every vertex of $V \setminus C$ is adjacent to a vertex of C. There are graphs which have no dominating cycles, and moreover, determining whether a graph has a dominating cycle on at most k vertex is NP–complete even for chordal, bipartite and split graphs [5, 7, 8].

The problems of finding $HCN(L(G))$ and a $MDTS$ on G are well known to be NP–hard [4]. Agnetis et al. [1] showed the NP–hardness of $MDTS$ when G is bipartite, and proposed a heuristic approach. Polynomial time algorithms for finding $HCN(L(G))$ and $MDTS$ have been found when G is a tree [2, 19], a cactus [9], and an AT-free graph[15]. The motivation for studying the $MDTS$ problem originates from applications in scheduling [1] and in other domains [11], such as data structures updating, genetics and combinatorial chemistry.

The paper is organized as follows. A lower bound for the $MDTS$ problem for general graphs is presented in Section 2. Upper bounds for the problem are presented in Section 3. Conclusions follow in Section 4.

2 A Lower Bound for the $MDTS$ problem

Throughout the paper let $G = (V, E)$ be a connected graph. We call $ad(i)$ the set of nodes adjacent to node i, and $\delta(i)$ the cardinality of $ad(i)$ (i.e., the degree of i). We refer to the notation proposed by Agnetis et al. [2], in which a marking function $\mu : V \to \{0, 1\}$ has been introduced. A node i such that $\mu(i) = 1$ is called *marked*. Marking a node means that at least one element of the trail set *must* pass through that node of the graph. In the following, the problem of finding a *minimum constrained dominating trail set* ($MCDTS$) is defined.

Definition 2.1 *Given a pair* (G, μ), *a* constrained dominating trail set Σ_c *is a collection of disjoint trails* $\{t_1, t_2, \ldots, t_r\}$ *such that: (1)* Σ_c *dominates all the edges of* G*; (2) for each marked node* i*, a trail* $t \in \Sigma_c$ *containing* i *exists. A* minimum constrained dominating trail set *(MCDTS) is a constrained dominating trail set of minimum cardinality.*

Finding $MDTS$ in G can therefore be reformulated as the problem of finding a $MCDTS$ on (G, μ), where G is the original graph, $\mu(i) = 0, \forall i \in V$. Let us recall the definitions of *bridge* and *cutvertex*. A bridge is an edge that does not lie on any cycle, and its removal produces a disconnected graph. A cutvertex is a node whose removal results in a disconnected graph. In the following some elementary graph transformations are presented.

Definition 2.2 *Given a pair* (G, μ)*, and a bridge* $b = (v_1, v_2)$*. Let* G_1 *and* G_2 *be the two subgraphs generated from* G *by the removal of the edge* b *(*$v_1 \in G_1$*,* $v_2 \in G_2$*). Let* G_1 *be composed by more than one node. By the* collapse of the terminal component G_1 *(with respect to the bridge* b*) we mean the transformation from* (G, μ) *to* $(\bar{G}, \bar{\mu})$ *defined as follows:* $\bar{G} = G_2 \cup b \cup v_1$*,* $\bar{\mu}(v_1) = 1$*, and* $\bar{\mu}(i) = \mu(i)$*,* $\forall i \in G_2$*.*

Note that if G_1 contains a single node, the conditions of Definition 2.2 are not satisfied. If it is possible to dominate the graph G_1 with a single trail entering in v_1, then the following theorem holds.

Theorem 2.3 *Given a pair* (G, μ)*, let* Σ_c^* *a* $MCDTS$*. Let* $\bar{G}$ *be the graph obtained applying the transformation collapse of the terminal component* G_1 *of* G *and let* $\bar{\Sigma}_c^*$ *be a* $MCDTS$ *on* $(\bar{G}, \bar{\mu})$*. If* G_1 *is dominated by a single trail* t_{G_1} *ending in* v_1 *then* $|\bar{\Sigma}_c^*| = |\Sigma_c^*|$*.*

Proof. Let $\Sigma_c^* = \{t_1, t_2, \ldots, t_r\}$. Two cases arise: (1) The bridge b belongs to a trail ($b \in t_h \subseteq \Sigma_c^*$), then $t_h = \{\ldots, v_2, b, v_1\} \cup t_{G_1}$; where t_{G_1} is the trail dominating G_1. Hence $\bar{\Sigma}_c^*$ can be obtained from Σ_c^* substituting $\bar{t}_h = \{\ldots, v_2, b, v_1\}$ for t_h. (2) The bridge b does not belong to any trail in Σ_c^*. Since in Σ_c^* a dominating trail t_h in G_1 there exists, then, by the hypothesis, it is always possible to substitute the trail t_h with one trail $\bar{t}_h$ dominating G_1 and ending in v_1. Therefore it is possible to add the bridge b to the trail $\bar{t}_h$, and using the same arguments of case (1), $\bar{\Sigma}_c^*$ can be obtained. Hence $|\bar{\Sigma}_c^*| = |\Sigma_c^*|$. □

A straightforward consequence of Definition 2.2 and Theorem 2.3 is the following Corollary.

Corollary 2.4 *Given a pair* (G, μ)*, let* Σ_c^* *a* $MCDTS$*. Let* $\bar{G}$ *be the graph obtained applying the transformation collapse of the terminal component* G_1 *of* G *and let* $\bar{\Sigma}_c^*$ *be a* $MCDTS$ *on* $(\bar{G}, \bar{\mu})$*. Then* $|\bar{\Sigma}_c^*| \leq |\Sigma_c^*|$*.*

Definition 2.5 *Let* (G, μ) *be a pair, and* $b_1 = (v_1, v_2)$ *and* $b_2 = (v_3, v_4)$ *be two bridges of* G *having no endvertices in common. Let* G_1*,* G_2 *and* G_3 *be the three subgraphs generated from* G *by the removal of the bridges* b_1 *and* b_2*, where* $v_1 \in G_1$*,* $v_2, v_3 \in G_2$ *and* $v_4 \in G_3$*. By the* collapse of the intermediate component G_2 *(with respect to bridges* b_1 *and* b_2*) we mean the transformation from* (G, μ) *to* $(\bar{G}, \bar{\mu})$ *in which* G_2 *is collapsed in a single marked node* v_x*. Hence* $\bar{G} = G_1 \cup G_3 \cup v_x \cup (v_1, v_x) \cup (v_x, v_3)$*,* $\bar{\mu}(v_x) = 1$ *and* $\bar{\mu}(i) = \mu(i)$*,* $\forall i \in G_1 \cup G_3$*.*

If the intermediate component G_2 can be dominated by a single trail starting in v_2 and ending in v_3 then Theorem 2.6 follows.

Theorem 2.6 *Given a pair (G, μ), let Σ_c^* a $MCDTS$. Let $(\bar{G}, \bar{\mu})$ be the transformed graph obtained applying the collapse of the intermediate component G_2 of G and let $\bar{\Sigma}_c^*$ be a $MCDTS$ on $(\bar{G}, \bar{\mu})$. If G_2 can be dominated by a single trail t_{G_2} starting in v_2 and ending in v_3, then $|\bar{\Sigma}_c^*| = |\Sigma_c^*|$.*

Proof. Observing that a trail t passing in v_2 and v_3 and dominating G_2 exists, using similar arguments of Theorem 2.3, $|\bar{\Sigma}_c^*| = |\Sigma_c^*|$ follows. □

As straightforward consequence of Definition 2.5 and Theorem 2.6 following Corollary holds.

Corollary 2.7 *Given a pair (G, μ), let Σ_c^* a $MCDTS$. Let $(\bar{G}, \bar{\mu})$ be the transformed graph obtained applying the collapse of the intermediate component G_2 of G and let $\bar{\Sigma}_c^*$ be a $MCDTS$ on $(\bar{G}, \bar{\mu})$. Then $|\bar{\Sigma}_c^*| \leq |\Sigma_c^*|$.*

In the following, let $(\bar{G} = (\bar{V}, \bar{E}), \bar{\mu})$ be the pair obtained from $(G = (V, E), \mu)$ by applying a sequence of collapse transformations, either terminal or intermediate components. Let G' be the graph obtained by removing all marked nodes (and the incident edges) from $\bar{G}$. Note that G' may contain more than one connected component. In the following we call these components *residual components* of $\bar{G}$. In Definition 2.8 a transformation for $\bar{G}$ is introduced.

Definition 2.8 *Let $(\bar{G}, \bar{\mu})$ be the pair obtained from (G, μ) by applying a sequence of transformations introduced in Definitions 2.2 and 2.5. Let $\{G'_1, \ldots, G'_k\}$ be the* residual components *of $\bar{G}$. By the* collapse of the residual components *we mean the transformation from $(\bar{G}, \bar{\mu})$ to the pair $(\tilde{G} = (\tilde{V}, \tilde{E}), \tilde{\mu})$ in which each residual component of $\bar{G}$ is shrunk in a single not marked node and $\tilde{\mu}(i) = \bar{\mu}(i) = 1$, $\forall i \in \{\tilde{G} \cap \bar{G}\}$.*

Theorem 2.9 *Let $(\tilde{G}, \tilde{\mu})$ be the pair obtained applying transformation* collapse of the residual components *to $(\bar{G}, \bar{\mu})$. Let $\bar{\Sigma}_c^*$ and $\tilde{\Sigma}_c^*$ be $MCDTSs$ on $(\bar{G}, \bar{\mu})$ and $(\tilde{G}, \tilde{\mu})$, respectively. Then $|\tilde{\Sigma}_c^*| \leq |\bar{\Sigma}_c^*|$.*

Proof. Let G'_h be a residual component of $\bar{G}$ and let E_h be the set of edges (i, j) in $\bar{G}$, such that $i \notin G'_h$ and $j \in G'_h$. Shrinking G'_h in a single not marked node implies that G'_h does not require to be dominated by any trail. Moreover, we assume that, for any pair of edges (i_1, j_1), (i_2, j_2) in E_h, a trail t_h passing in i_1 and i_2 and dominating all edges in G'_h exists. The above observations can be applied to all the residual components. Since such trails t_h, $h = 1, \ldots, k$, may not exist, then $|\tilde{\Sigma}_c^*| \leq |\bar{\Sigma}_c^*|$ and the thesis follows. □

Note that graph $\tilde{G}$ has a particular structure. In fact, the transformation collapse of a terminal component produces a marked leaf and the transformation collapse of an intermediate component produces a marked cutvertex of degree 2. Whereas the transformation collapse of the residual components generates the only nodes in $\tilde{G}$ that may have degree greater than 2. By construction, such nodes are always adjacent to the marked nodes generated by transformations of Definitions 2.2 and 2.5. Since the marked nodes are either leaves or cutvertices, Lemma 2.10 follows.

Lemma 2.10 *The graph $\tilde{G}$ is a tree.*

A $MCDTS$ on the pair $(\tilde{G}, \tilde{\mu})$ can be computed in $O(|\tilde{V}|)$ applying the algorithm DOMTREE [2]. Moreover, the following theorem provides the $|\tilde{\Sigma}_c^*|$ in closed form.

Theorem 2.11 *Let $\tilde{\Sigma}_c^*$ be a $MCDTS$ on $\tilde{G}$. Then $|\tilde{\Sigma}_c^*| = \max\{\lceil \frac{\tilde{N}_{odd}}{2} \rceil, 1\}$ where $\tilde{N}_{odd}$ is the number of nodes with odd degree in $\tilde{G}$.*

Proof. Let i be a node with odd degree in $\tilde{G}$. Two cases arise: (1) i is marked; (2) i is not marked. In Case (1), by construction i is a leaf and, hence, there exists a trail in $\tilde{\Sigma}_c^*$ ending in i. In Case (2), since all adjacent nodes of i are marked, and since i has odd degree, it is easy to see that exactly $\lfloor \frac{\delta(i)}{2} \rfloor$ trails in $\tilde{\Sigma}^*$ pass through node i and only one trail must end either in i or in a node in $ad(i)$. In fact, let us suppose that two trails, say $t_1 = \{v_1, \ldots, s\}$ and $t_2 = \{v_2, \ldots, r\}$, with $r, s \in ad(i) \cup \{i\}$ exists in $\tilde{\Sigma}^*$. Obviously, trails t_1 and t_2 can be joined in the trail $t_3 = \{v_1, \ldots, s, i, r, \ldots, v_2\}$. By removing t_1 and t_2 from $\tilde{\Sigma}_c^*$ and adding t_3, a $MCDTS$ of cardinality $|\tilde{\Sigma}_c^*| - 1$ is obtained, contradicting the hypothesis. Thus for each node $i \in \tilde{G}$ of odd degree there exists exactly one trail of $\tilde{\Sigma}^*$ ending in i or in a node of $ad(i)$. Note that if a trail ending in a node of $ad(i)$ exists, then it is always possible to replace such trail with a trail ending in i. Hence, using arguments similar to case (2), it is always possible to build a $MCDTS$ in which no trail ends in a node of even degree in $\tilde{G}$.
Since trails of $\tilde{\Sigma}_c^*$ start and end either in nodes of odd degree or in nodes adjacent to them, then $|\tilde{\Sigma}_c^*| = \max\{\lceil \frac{\tilde{N}_{odd}}{2} \rceil, 1\}$. When $N_{odd} = 0$, $\tilde{G}$ is eulerian and $|\tilde{\Sigma}_c^*| = 1$. □

By Theorems 2.9 and 2.11 follows the Corollary 2.12 that provides a lower bound for a $MDTS$ on the original graph G.

Corollary 2.12 *Let Σ^* be a $MDTS$ on G. Then $|\Sigma^*| \geq \max\{\lceil \frac{\tilde{N}_{odd}}{2} \rceil, 1\}$ where $\tilde{N}_{odd}$ is the number of nodes with odd degree of graph $\tilde{G}$ obtained from G.*

2.1 Computing the Lower Bound

In this section, we describe a procedure to compute the lower bound provided by the Corollary 2.12.

1. The algorithm applies to all the terminal components that are either an edge or that are composed of more than two nodes without containing bridges, the transformation of Definition 2.2.

2. The algorithm applies to all the intermediate components that are either an edge or that are composed of more than two nodes without containing bridges, the transformation of Definition 2.5.

3. Finally the tranformation of Definition 2.8 is applied to the resulting graph, and a lower bound is obtained by applying Corollary 2.12 on $\tilde{G}$.

Let LB be the bound achieved by the above procedure.
Note that, when the transformation of Definition 2.2 is applied to a component containing a bridge it does not improve the lower bound. In fact if G_1 contains q marked leaves at

least $\lceil q/2 \rceil$ trails are required to dominate G_1; hence if G_1 is collapsed in a single marked node we are implicitly assuming that G_1 can be dominated by a single trail. Similarly, it is not possible to improve the lower bound by applying transformation of Definition 2.5 to a component containing a bridge.
On the other hand, transformation of Definition 2.2 introduces a marked node of odd degree, therefore the best lower bound is achieved by applying it to all the terminal components that are either an edge or that are composed of more than two nodes without containing bridges.
Moreover the transformation of Definition 2.5 introduces a marked node of degree two and two nodes adjacent to it of degree δ_1 and δ_2 in $\tilde{G}$. Whereas by not applying it only a node of degree $\delta_1 + \delta_2 - 2$ is produced in $\tilde{G}$.
Hence, the best value of the lower bound is obtained when transformation of Definition 2.5 is applied to all the intermediate components composed of either an edge or more than two nodes without containing bridges.
Since all transformations of Definition 2.2 and 2.5 are applied to different and disjoint components, and since no bridges are produced by these transformations, then the order in which the transformations are performed does not affect the lower bound value. In other words any order of transformations produces the same graph $\tilde{G}$.
From the above discussion it follows that no procedure relying only on transformations of Definitions 2.2, 2.5 and 2.8 may lead to a better lower bound.

Since all the bridges and the components can be determined in $O(|V| + |E|)$ using a Modified Depth First Search algorithm [20] the lower bound procedure can be implemented to run in linear time using an adequate data structure.

3 Upper Bounds for the $MDTS$ problem

In the aim of this section, since the $MDTS$ problem is NP–hard for general graphs, may be interesting to determine upper bounds on the value of $|\Sigma^*|$. Theorems 3.1 and 3.2, proposed by Bollobás [6], yield two simple upper bounds on the value of $|\Sigma^*|$ indicated as UB_1 and UB_2, respectively.

Theorem 3.1 *A graph with $n = |V|$ nodes and at most one vertex of even degree can be covered by $\lfloor n/2 \rfloor$ edge disjoint paths.*

Theorem 3.2 *A graph G with $n_{even} \geq 1$ vertices of even degree and n_{odd} vertices of odd degree can be covered by $n_{even} + (n_{odd}/2) - 1$ edge disjoint paths.*

Moreover, these paths can be chosen in such a way that each vertex of even degree, with the exception of exactly one, is the endvertex of exactly two paths.

Agnetis et al. [1] proposed a heuristic for the $MDTS$ problem. In the aim of this paper, their algorithm, based on the concept of odd vertex pairing (OVP), yields another simple bound for $|\Sigma^*|$ (in what follows indicated as UB_{OVP}), improving bounds UB_1 and UB_2. In [1] the following result is presented.

Theorem 3.3 *Let G be a connected graph with N_{odd} vertices with odd degree, then $|\Sigma^*| \leq \max\{\lceil N_{odd}/2 \rceil, 1\}$.*

A way to obtain an upper bound on $|\Sigma^*|$ refers to the concept of split graphs [18]. A graph G is a split graph if the set of nodes can be partitioned into a clique K and an independent set I. In particular, considering a connected split graph G, every cycle spanning the clique K obviously dominates vertices contained either in K and in the independent set I. An upper bound on $|\Sigma^*|$ for a graph G may be obtained calculating the minimum number of edges to be added (or erased) in the graph G to make it a split graph. In order to proof this bound it is useful to recall the following definition of splittance.

Definition 3.4 *The splittance of an arbitrary undirected graph G is the minimum number of edges to be added or erased in order to obtain a split graph.*

Split graphs are graphs with splittance zero. The splittance of a graph can be computed using the following Theorem 3.5 due to Hammer and Simeone [13].

Theorem 3.5 *Let $G = (V, E)$ be an undirected graph with degree sequence $\delta(1) \geq \delta(2) \geq \ldots \geq \delta(|V|)$, and let $m = max\{i \mid \delta(i) \geq i - 1\}$. Then, the splittance $\Lambda(G)$ of the graph G is given by the expression $\frac{1}{2}[m(m-1) + \sum_{i=m+1}^{|V|} \delta(i) - \sum_{i=1}^{m} \delta(i)]$.*

The proposed bound on $|\Sigma^*|$, indicated as $UB.s$, is now presented more formally.

Theorem 3.6 *Let $G = (V, E)$ be a connected graph with splittance $\Lambda(G)$, then $|\Sigma^*| \leq \max\{\Lambda(G) - 1, 1\}$.*

Proof. By contradiction let $|\Sigma^*| > \Lambda(G)$. If $\Lambda(G) > 1$ there are $\Lambda(G)$ edges to be added or removed from G to obtain a split graph $S_G = (V', E')$. $V' = V(K) \cup V(I)$, where $V(K)$ and V(I) are the node set of the clique and the independent set, respectively, in which we can partition V. $E' = E \cup E^+ \setminus E^-$, where E^+ is the set of edges to be added and E^- is the set of edges to be removed from G. Every edge $e \in E^-$ may be viewed as a trail on G (restricted to one only edge) that dominates e and every edge adjacent to it. Every edge $e \in E^+$ will be dominated in the split graph S_G by a path contained in the clique. Let us consider the dominating path t of S_G all contained in the clique and with the minimum number ν of edges in E^+. When the edges E^+ are removed from t, $\nu - 1$ paths are obtained dominating all the edges $E \setminus E^-$ in G. Thus, we have in G a dominating path set of cardinality at most $|E^-| + |E^+| - 1$, and a contradiction is found. If $\Lambda(G) \leq 1$, since split graphs have $|\Sigma^*| = 1$, the theorem holds in view of the definition of splittance. □

Another way to obtain upper bounds on $|\Sigma^*|$ refers to the concept of interval graphs. An intersection representation of a graph G assigns each vertex x a set $s(x)$ such that x, y are adjacent if and only if $s(x) \cap s(y) \neq \emptyset$. On the other hand, the graph G is the intersection graph of the sets in the representation. Interval graphs are the intersection graphs obtainable by assigning each vertex to a single interval on the real line. An intersection representation s that assigns to each vertex a union of intervals on the real line is a multiple–interval representation of G. Let $|s(x)|$ be the number of pairwise disjoint intervals whose union is $s(x)$. If $|s(x)| = k$, then $s(x)$ consists of k intervals (or x is assigned k intervals).
Two parameters have been introduced in order to measure how far is a graph from being

an interval graph. The *interval number* of G is $IN(G) = min_s max_{x \in V}|s(x)|$, in which the minimum is considered over all multiple–interval representations of G. The *total interval number* of G is $TIN(G) = min_s \sum_{x \in V}|s(x)|$. Clearly $TIN(G) \leq |V|IN(G)$ and a connected interval graph has $IN(G) = 1$ and $TIN(G) = |V|$. In particular, it is useful to recall the following result [3, 17].

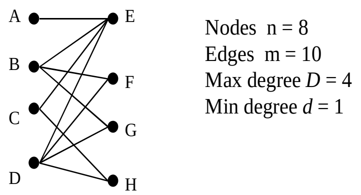
Theorem 3.7 *If $G(V,E)$ is a connected graph, then $TIN(G) \leq |E| + |\Sigma^*|$, moreover if G is also triangle–free then $TIN(G) = |E| + |\Sigma^*|$.*

Hence, in the case of triangle–free graphs, every upper bound for $TIN(G)$ provides a simple way to determine an upper bound for $|\Sigma^*|$ according to Theorem 3.7. In view of Theorem 3.7, the following upper bounds result from works on extremal values of the $TIN(G)$ [3, 12, 16, 17].

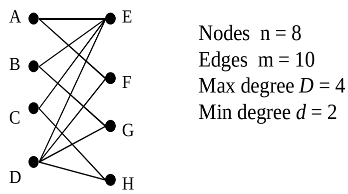
Theorem 3.8 *Let $G(V,E)$ be a connected and triangle–free graph with $m = |E|$, $n = |V|$, maximum degree D and minimum degree d, then the following upper bounds (UB.a–UB.e) hold for $|\Sigma^*|$:*

UB.a	UB.b	UB.c	UB.d	UB.e
$\max\{\lceil \frac{(n^2+1)}{4} \rceil - m, 1\}$	$\lfloor \frac{(5m+2)}{4} \rfloor - m$	$\lfloor \frac{(9m+1)}{8} \rfloor - m$ if $d \geq 2$	$\max\{\lfloor (D + \frac{1}{D}) \frac{n}{2} \rfloor - m, 1\}$	$\max\{\lfloor \frac{Dn}{2} \rfloor + 1 - m, 1\}$ if D is even

Note that, in the case of triangle-free graphs the bounds $UB.s$ and UB_{OVP}, based on the splittance and the odd vertex pairing respectively, result not dominated by those based on $TIN(G)$ as shown in examples reported in Figure 1.



UB	UB.a	UB.b	UB.c	UB.d	UB.e	UB.s	UB_{OVP}	UB_1	UB_2
Value	7	3	-	7	7	1	1	4	6



UB	UB.a	UB.b	UB.c	UB.d	UB.e	UB.s	UB_{OVP}	UB_1	UB_2
Value	7	3	1	7	7	2	1	4	7

Figure 1: Upper bounds for triangle-free graphs: an example

4 Conclusions

In this paper, lower and upper bounds for the problem of finding a dominating trail set of minimum cardinality are presented. In particular, a lower bound is obtained applying simple graph transformations and upper bounds can be computed in closed form. Such bounds may be useful in algorithm design and test as well as in graph design when a limited cardinality of the $MDTS$ is required.
The propoesed procedure is simple and it produces good quality bounds, in fact it implements a generalization of a simpler bound used to prove the optimality of the solutions found by a metaheuristic algorithm [10] on more than 98% of the cases on a wide set of randomly generated instances.

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